

CONVERGENCE IN HIGHER MEAN OF A RANDOM SCHRÖDINGER TO A LINEAR BOLTZMANN EVOLUTION

THOMAS CHEN

ABSTRACT. We study the macroscopic scaling and weak coupling limit for a random Schrödinger equation on \mathbb{Z}^3 . We prove that the Wigner transforms of a large class of "macroscopic" solutions converge in r -th mean to solutions of a linear Boltzmann equation, for any finite value of $r \in \mathbb{R}_+$. This extends previous results where convergence in expectation was established.

1. INTRODUCTION

We study the macroscopic scaling and weak coupling limit of the quantum dynamics in the three dimensional Anderson model, generated by the Hamiltonian

$$H_\omega = -\frac{1}{2}\Delta + \lambda V_\omega(x) \quad (1)$$

on $\ell^2(\mathbb{Z}^3)$. Here, Δ is the nearest neighbor discrete Laplacian, $0 < \lambda \ll 1$ is a small coupling constant that defines the disorder strength, and the random potential is given by $V_\omega(x) = \omega_x$, where $\{\omega_x\}_{x \in \mathbb{Z}^3}$, are independent, identically distributed Gaussian random variables.

While the phenomenon of impurity-induced insulation is, for strong disorders $\lambda \gg 1$ or extreme energies, mathematically well understood (Anderson localization, [1, 6]), establishing the existence of electric conduction in the weak coupling regime $\lambda \ll 1$ is a key open problem of outstanding difficulty. A particular strategy to elucidate aspects of the latter, which has led to important recent successes (especially [5]), is to analyze the macroscopic transport properties derived from the microscopic quantum dynamics generated by (1), [3, 4, 5, 9, 2].

Let $\phi_t \in \ell^2(\mathbb{Z}^3)$ be the solution of the random Schrödinger equation

$$\begin{cases} i\partial_t \phi_t &= H_\omega \phi_t \\ \phi_0 &\in \ell^2(\mathbb{Z}^3), \end{cases} \quad (2)$$

with a deterministic initial condition ϕ_0 which is supported on a region of diameter $O(\lambda^{-2})$. Let $W_{\phi_t}(x, v)$ denote its *Wigner transform*, where $x \in \frac{1}{2}\mathbb{Z}^3 \equiv (\mathbb{Z}/2)^3$, and $v \in \mathbb{T}^3 = [0, 1]^3$. We consider a scaling for small λ defined by the macroscopic time, position, and velocity variables $(T, X) := \lambda^2(t, x)$, $V := v$, while (t, x, v) are the microscopic variables. Likewise, we introduce an appropriately rescaled, macroscopic counterpart $W_\lambda^{resc}(T, X, V)$ of $W_{\phi_t}(x, v)$.

It was proved by Erdős and Yau for the continuum, [4, 3], and by the author for the lattice model, [2], that for any test function $J(X, V)$, and globally in macroscopic time T ,

$$\begin{aligned} \lim_{\lambda \rightarrow 0} \mathbb{E} \left[\int dX dV J(X, V) W_{\lambda}^{resc}(T, X, V) \right] \\ = \int dX dV J(X, V) F(T, X, V) , \end{aligned}$$

where $F(T, X, V)$ is the solution of a linear Boltzmann equation. For the random wave equation, a similar result is proved by Lukkarinen and Spohn, [7]. The corresponding local in T result was established much earlier by Spohn, [9].

The main goal of this paper is to improve the mode of convergence. We establish convergence in r -th mean,

$$\begin{aligned} \lim_{\lambda \rightarrow 0} \mathbb{E} \left[\left| \int dX dV J(X, V) W_{\lambda}^{resc}(T, X, V) \right. \right. \\ \left. \left. - \int dX dV J(X, V) F(T, X, V) \right|^r \right] = 0 , \quad (3) \end{aligned}$$

for any $r \in 2\mathbb{N}$, and thus for any finite $r \in \mathbb{R}_+$. Thus, in particular, we observe that the variance of $\int dX dV J(X, V) W_{\lambda}^{resc}(T, X, V)$ vanishes in this macroscopic, hydrodynamic limit.

Our proof comprises generalizations and extensions of the graph expansion methods introduced by Erdős and Yau in [4, 3], and further elaborated on in [2]. The structure of the graphs entering the problem is significantly more complicated than in [4, 3, 2], and the number of graphs in the expansion grows much faster than in [4, 3, 2] (superfactorial versus factorial). A main technical result in this paper establishes that the associated Feynman amplitudes are sufficiently small to compensate for the large number of graphs, which is shown to imply (3). This is similar to the approach in [4, 3, 2].

The present work addresses a time scale of order $O(\lambda^{-2})$ (as in [4, 3, 2]), in which the average number of collisions experienced by the electron is finite, so that ballistic behavior is observed. Accordingly, the macroscopic dynamics is governed by a linear Boltzmann equation. Beyond this time scale, the average number of collisions is *infinite*, and the level of difficulty of the problem increases drastically. In their recent breakthrough result, Erdős, Salmhofer and Yau have established that over a time scale of order $O(\lambda^{-2-\kappa})$ for an explicit numerical value of $\kappa > 0$, the macroscopic dynamics in $d = 3$ derived from the quantum dynamics is determined by a *diffusion equation*, [5].

We note that control of the macroscopic dynamics up to a time scale $O(\lambda^{-2})$ produces lower bounds of the same order (up to logarithmic corrections) on the localization lengths of eigenvectors of H_{ω} , see [2] for $d = 3$ (the same arguments are valid for $d \geq 3$). This extends recent results of Schlag, Shubin and Wolff, [8], who derived similar lower bounds for the weakly disordered Anderson model in dimensions $d = 1, 2$ using harmonic analysis techniques.

This work comprises a partial joint result with Laszlo Erdős (Lemma 5.2), to whom the author is deeply grateful for his support and generosity.

2. DEFINITION OF THE MODEL AND STATEMENT OF MAIN RESULTS

To give a mathematically well-defined meaning to all quantities occurring in our analysis, we first introduce our model on a finite box

$$\Lambda_L = \{-L, -L+1, \dots, -1, 0, 1, \dots, L-1, L\}^3 \subset \mathbb{Z}^3, \quad (4)$$

for $L \in \mathbb{N}$ much larger than any relevant scale of the problem, and take the limit $L \rightarrow \infty$ later. All estimates derived in the sequel will be uniform in L . We consider the discrete Schrödinger operator

$$H_\omega = -\frac{1}{2} \Delta + \lambda V_\omega \quad (5)$$

on $\ell^2(\Lambda_L)$ with periodic boundary conditions. Here, Δ is the nearest neighbor Laplacian,

$$(\Delta f)(x) = 6f(x) - \sum_{|y-x|=1} f(y), \quad (6)$$

and

$$V_\omega(x) = \omega_x \quad (7)$$

is a random potential with $\{\omega_y\}_{y \in \Lambda_L}$ i.i.d. Gaussian random variables satisfying $\mathbb{E}[\omega_x] = 0$, $\mathbb{E}[\omega_x^2] = 1$, for all $x \in \Lambda_L$. Expectations of higher powers of ω_x satisfy Wick's theorem, cf. [4], and our discussion below. Clearly, $\|V_\omega\|_{\ell^\infty(\Lambda_L)} < \infty$ almost surely (a.s.), and H_ω is a.s. self-adjoint on $\ell^2(\Lambda_L)$, for every $L < \infty$.

Let $\Lambda_L^* = \frac{1}{L} \Lambda_L = \{-1, -\frac{L-1}{L}, \dots, -\frac{1}{L}, 0, \frac{1}{L}, \dots, \frac{L-1}{L}, 1\}^3 \subset \mathbb{T}^3$ denote the lattice dual to Λ_L , where $\mathbb{T}^3 = [-1, 1]^3$ the 3-dimensional unit torus. For $0 < \rho \leq 1$ with $\frac{1}{\rho} \in \mathbb{N}$, we define $\Lambda_{L,\rho} := \rho \Lambda_{\rho^{-1}L}$, and note that its dual lattice is given by $\Lambda_{L,\rho}^* = \frac{1}{L} \Lambda_{\rho^{-1}L} \subset \rho^{-1} \mathbb{T}^3$. For notational convenience, we shall write $\int_{\Lambda_{L,\rho}} dk \equiv \sum_{k \in \Lambda_{L,\rho}}$, and $\int_{\rho^{-1}\mathbb{T}^3} dk$ for the Lebesgue integral. For the Fourier transform and its inverse, we use the convention

$$\widehat{f}(k) = \rho^3 \sum_{x \in \Lambda_{L,\rho}} e^{-2\pi i k \cdot x} f(x), \quad g^\vee(x) = \int_{\Lambda_{L,\rho}^*} dk g(k) e^{2\pi i k \cdot x}, \quad (8)$$

for $L \leq \infty$ (where $\Lambda_{\infty,\rho} = \rho \mathbb{Z}^3$ and $\Lambda_{\infty,\rho}^* = \mathbb{T}^3/\rho$). We will mostly use $\rho = 1$, and sometimes $\rho = \frac{1}{2}$. On $\Lambda_{L,\rho}^*$, we define $\delta(k) = \widehat{1}(k)$ with $\delta(0) = |\Lambda_{L,\rho}|$ if $k = 0$ and $\delta(k) = 0$ if $k \neq 0$. On \mathbb{T}^d or \mathbb{R}^d , δ will denote the usual d -dimensional delta distribution. The nearest neighbor lattice Laplacian defines the Fourier multiplier

$$(-\Delta f)^\wedge(k) = 2e_\Delta(k) \widehat{f}(k), \quad (9)$$

where

$$e_\Delta(k) = \sum_{i=1}^3 (1 - \cos(2\pi k_i)) = 2 \sum_{i=1}^3 \sin^2(\pi k_i) \quad (10)$$

determines the kinetic energy of the electron.

Let $\phi_t \in \ell^2(\Lambda_L)$ denote the solution of the random Schrödinger equation

$$\begin{cases} i\partial_t \phi_t &= H_\omega \phi_t \\ \phi_0 &\in \ell^2(\Lambda_L), \end{cases} \quad (11)$$

for a fixed realization of the random potential. We define its (real, but not necessarily positive) Wigner transform $W_{\phi_t} : \Lambda_{L, \frac{1}{2}} \times \Lambda_L^* \rightarrow \mathbb{R}$ by

$$W_{\phi_t}(x, v) := 8 \sum_{\substack{y, z \in \Lambda_L \\ y+z=2x}} \overline{\phi_t(y)} \phi_t(z) e^{2\pi i(y-z)v} . \quad (12)$$

Fourier transformation with respect to the variable $x \in \Lambda_{L, \frac{1}{2}}$ (i.e. (8) with $\rho = \frac{1}{2}$, see [5] for more details) yields

$$\widehat{W}_{\phi_t}(\xi, v) = \overline{\widehat{\phi}_t(v - \frac{\xi}{2})} \widehat{\phi}_t(v + \frac{\xi}{2}) , \quad (13)$$

for $v \in \Lambda_L^*$ and $\xi \in \Lambda_{L, \frac{1}{2}}^* \subset 2\mathbb{T}^3$.

The Wigner transform is the key tool in our derivation of the macroscopic limit for the quantum dynamics described by (19). For $\eta > 0$ small, we introduce macroscopic variables $T := \eta t$, $X := \eta x$, $V := v$, and consider the rescaled Wigner transform

$$W_{\phi_t}^\eta(X, V) := \eta^{-3} W_{\phi_t}(\eta^{-1}X, V) \quad (14)$$

for $T \geq 0$, $X \in \eta\Lambda_{L, \frac{1}{2}}$, and $V \in \Lambda_L^*$.

For a Schwartz class function $J \in \mathcal{S}(\mathbb{R}^3 \times \mathbb{T}^3)$, we write

$$\langle J, W_{\phi_t}^\eta \rangle := \sum_{X \in \eta\Lambda_{L, \frac{1}{2}}} \int_{\Lambda_L^*} dV \overline{J(X, V)} W_{\phi_t}^\eta(X, V) . \quad (15)$$

With \widehat{W}_{ϕ_t} as in (13), we have

$$\langle J, W_{\phi_t}^\eta \rangle = \langle \widehat{J}_\eta, \widehat{W}_{\phi_t} \rangle = \int_{\Lambda_{L, \frac{1}{2}}^* \times \Lambda_L^*} d\xi dv \overline{\widehat{J}_\eta(\xi, v)} \widehat{W}_{\phi_t}(\xi, v) , \quad (16)$$

where $J_\eta(x, v) := \eta^{-3} J(\eta x, v)$, and

$$\widehat{J}_\eta(\xi, v) = \eta^{-3} \sum_{x \in \Lambda_{L, \frac{1}{2}}} J(\eta x, v) e^{-2\pi i x \xi} = \eta^{-3} \sum_{X \in \eta\Lambda_{L, \frac{1}{2}}} J(X, v) e^{-\frac{2\pi i X \xi}{\eta}} . \quad (17)$$

We note that in the limit $L \rightarrow \infty$, $\widehat{J}_\eta(\xi, v)$ tends to a smooth delta function with respect to the ξ -variable, of width $O(\eta)$ and amplitude $O(\eta^{-1})$, but remains uniformly bounded with respect to η in the v -variable.

The macroscopic scaling limit obtained from letting $\eta \rightarrow 0$, with $\eta = \lambda^2$, is determined by a linear Boltzmann equation. This was proven in [2] for \mathbb{Z}^3 , and non-Gaussian distributed random potentials (the Gaussian case follows also from [2]). The corresponding result for the continuum model in dimensions 2, 3 was proven in [4].

Theorem 2.1. *For $\eta > 0$, let*

$$\phi_0^\eta(x) := \eta^{\frac{3}{2}} \frac{h(\eta x) e^{2\pi i \frac{S(\eta x)}{\eta}}}{\|h\|_{\ell^2(\eta\mathbb{Z}^3)}} , \quad (18)$$

with $h, S \in \mathcal{S}(\mathbb{R}^3, \mathbb{R})$ of Schwartz class, and $\|h\|_{L^2(\mathbb{R}^3)} = 1$. Assume L sufficiently large (see (73)) that $\phi_0^\eta|_{\Lambda_L} = \phi_0^\eta$. Let ϕ_t^η be the solution of the random Schrödinger equation

$$i \partial_t \phi_t^\eta = H_\omega \phi_t^\eta \quad (19)$$

on $\ell^2(\Lambda_L)$ with initial condition ϕ_0^η , and let

$$W_T^{(\eta)}(X, V) := W_{\phi_{\eta^{-1}T}^\eta}^\eta(X, V) \quad (20)$$

denote the corresponding rescaled Wigner transform.

Choosing

$$\eta = \lambda^2, \quad (21)$$

where λ is the coupling constant in (5), it follows that

$$\lim_{\lambda \rightarrow 0} \lim_{L \rightarrow \infty} \mathbb{E}[\langle J, W_T^{(\lambda^2)} \rangle] = \langle J, F_T \rangle, \quad (22)$$

where $F_T(X, V)$ solves the linear Boltzmann equation

$$\begin{aligned} \partial_T F_T(X, V) + \sum_{j=1}^3 (\sin 2\pi V_j) \partial_{X_j} F_T(X, V) \\ = \int_{\mathbb{T}^3} dU \sigma(U, V) [F_T(X, U) - F_T(X, V)] \end{aligned} \quad (23)$$

with initial condition

$$\begin{aligned} F_0(X, V) &= w - \lim_{\eta \rightarrow 0} W_{\phi_0^\eta}^\eta \\ &= |h(X)|^2 \delta(V - \nabla S(X)), \end{aligned} \quad (24)$$

and

$$\sigma(U, V) := 2\pi \delta(e_\Delta(U) - e_\Delta(V))$$

denotes the collision kernel.

The purpose of the present work is to obtain a significant improvement of the mode of convergence.

Our main result is the following theorem.

Theorem 2.2. Assume that the Fourier transform of (18), $\widehat{\phi_0^\eta}$, satisfies the concentration of singularity property (29) - (31). Then, for any fixed, finite $r \in 2\mathbb{N}$, any $T > 0$, and for any Schwartz class function J , the estimate

$$\lim_{L \rightarrow \infty} \left(\mathbb{E} \left[\left| \langle J, W_T^{(\lambda^2)} \rangle - \mathbb{E}[\langle J, W_T^{(\lambda^2)} \rangle] \right|^r \right] \right)^{\frac{1}{r}} \leq c(r, T) \lambda^{\frac{1}{300r}} \quad (25)$$

holds for λ sufficiently small, and a finite constant $c(r, T)$ that does not depend on λ . Consequently,

$$\lim_{\lambda \rightarrow 0} \lim_{L \rightarrow \infty} \mathbb{E} \left[\left| \langle J, W_T^{(\lambda^2)} \rangle - \langle J, F_T \rangle \right|^r \right] = 0 \quad (26)$$

(i.e. convergence in r -th mean), for any finite $r, T \in \mathbb{R}_+$.

We observe that, in particular, the variance of $\langle J, W_T^{(\lambda^2)} \rangle$ vanishes in the limit $\lambda \rightarrow 0$. Moreover, the following result is an immediate consequence.

Corollary 2.1. *Under the assumptions of Theorem 2.2, the rescaled Wigner transform $W_T^{(\lambda^2)}$ convergences weakly, and in probability, to a solution of the linear Boltzmann equations, globally in $T > 0$, as $\lambda \rightarrow 0$. That is, for any finite $T > 0$, any $\nu > 0$, and any J of Schwartz class,*

$$\mathbb{P} \left[\lim_{\lambda \rightarrow 0} \left| \langle J, W_T^{(\lambda^2)} \rangle - \langle J, F_T \rangle \right| > \nu \right] = 0, \quad (27)$$

where F_T solves (23) with initial condition (24).

2.1. Singularities of $\widehat{\phi}_0^\eta$. One obtains a well-defined semiclassical initial condition (24) for the linear Boltzmann evolution (23) if the initial condition is of WKB type (18), but in general not if the initial condition is only required to be in $\ell^2(\mathbb{Z}^3)$. However, for the expected value of the quantum fluctuations in (22) to converge to zero as $\lambda \rightarrow 0$, it is sufficient to have initial data in $\ell^2(\mathbb{Z}^3)$, see [4, 2].

As we will see, a key point in proving that as $\lambda \rightarrow 0$, the quantum fluctuations vanish in higher mean, i.e. (26), it is necessary to control the overlap of the singularities of $\widehat{\phi}_0^\eta$ with those of the resolvent multipliers $(e_\Delta(k) - \alpha \pm i\varepsilon)^{-1}$, where $\alpha \in \mathbb{R}$ and $\varepsilon = O(\eta) \ll 1$. As opposed to the case in (22), it cannot be expected that the quantum fluctuations vanish in higher mean for general L^2 initial data (for (22), the overlap of the singularities of $\widehat{\phi}_0^\eta$ and of those of the resolvent multipliers plays no rôle). Moreover, we note that the singularities of the WKB initial condition

$$\begin{aligned} \widehat{\phi}_0^\eta(k) &= \eta^{\frac{3}{2}} \sum_{x \in \mathbb{Z}^3} h(\eta x) e^{2\pi i \left(\frac{S(\eta x)}{\eta} - kx \right)} \\ &= \eta^{\frac{3}{2}} \sum_{X \in \eta \mathbb{Z}^3} h(X) e^{2\pi i \left(\frac{S(X) - kX}{\eta} \right)} \end{aligned} \quad (28)$$

(which are determined by the zeros of $\det \text{Hess } S(X)$, the determinant of the Hessian of S) will possess a rather arbitrary structure for generic choices of $S \in \mathcal{S}(\mathbb{R}^3, \mathbb{R})$. At present, we do not know if for WKB initial data of the form (18), the quantum fluctuations would converge to zero in higher mean without any further restrictions on the phase function $S \in \mathcal{S}(\mathbb{R}^3, \mathbb{R})$. A more detailed analysis of these questions is left for future work. In this paper, we shall assume that the Fourier transform of the WKB initial condition (18) satisfies a *concentration of singularity condition*:

$$\widehat{\phi}_0^\eta(k) = f_\infty^\eta(k) + f_{sing}^\eta(k), \quad (29)$$

where

$$\|f_\infty^\eta\|_{L^\infty(\mathbb{T}^3)} < c, \quad (30)$$

and

$$\| |f_{sing}^\eta| * |f_{sing}^\eta| \|_{L^2(\mathbb{T}^3)} = \| |f_{sing}^\eta|^\vee \|_{\ell^4(\mathbb{Z}^3)}^2 \leq c' \eta^{\frac{4}{5}} \quad (31)$$

for finite, positive constants c, c' independent of η . This condition imposes a restriction on the possible choices of the phase function S .

The following simple, but physically important examples of $\widehat{\phi}_0^\eta$ satisfy (29) - (31).

2.1.1. *Example.* Let $S(X) = pX$ for $X \in \text{supp}\{h\}$, and $p \in \mathbb{T}^3$. Then,

$$\widehat{\phi}_0^\eta(k) = \frac{\eta^{-\frac{3}{2}} \widehat{h}(\eta^{-1}(k-p))}{\|h\|_{\ell^2(\eta\mathbb{Z}^3)}} =: \delta_\eta(k-p). \quad (32)$$

Since h is of Schwartz class, δ_η is a smooth bump function concentrated on a ball of radius $O(\eta)$, with $\|\delta_\eta\|_{L^2(\mathbb{T}^3)} = 1$. Accordingly, we find

$$(|\delta_\eta| * |\delta_\eta|)(k) \approx \chi(|k| < c\eta), \quad (33)$$

and

$$\| |\delta_\eta| * |\delta_\eta| \|_{L^2(\mathbb{T}^3)} = \| |\delta_\eta|^\vee \|_{\ell^4(\mathbb{Z}^3)}^2 \leq c\eta^{\frac{3}{2}}. \quad (34)$$

Hence, (29) - (31) is satisfied, with $f_\infty^\eta = 0$. We remark that in this example, $p \in \mathbb{T}^3$ corresponds to the velocity of the macroscopic initial condition $F_0(X, V)$ in (24) for the linear Boltzmann evolution.

2.1.2. *Example.* As a small generalization of the previous case, we may likewise assume for S that for every $k \in \mathbb{T}^3$, there are finitely many solutions $X_j(k)$ of $\nabla_X S(X_j(k)) = k$, and that $X_j(\cdot) \in C^1(\text{supp}\{h\})$ for each j . Moreover, we assume that $|\det \text{Hess } S(X)| > c$ uniformly on $\text{supp}\{h\}$. Then, by stationary phase arguments, [10], one finds that

$$\widehat{\phi}_0^\eta(k) = f_\infty^\eta(k) + f_{\text{sing}}^\eta(k) \quad , \quad \|f_\infty^\eta\|_{L^\infty(\mathbb{T}^3)} < c \quad (35)$$

with

$$f_{\text{sing}}^\eta(k) = \sum_j c_j \delta_\eta^{(j)}(k - \nabla_X S(X_j(k))), \quad (36)$$

for constants c_j independent of η , and smooth bump functions $\delta_\eta^{(j)}$ similar to (32). One again obtains $\| |f_{\text{sing}}^\eta|^\vee \|_{\ell^4(\mathbb{Z}^3)}^2 \leq c\eta^{\frac{3}{2}}$, which verifies that (29) - (31) holds. ∇S determines the velocity distribution of the macroscopic initial condition $F_0(X, V)$ in (24).

3. PROOF OF THEOREM 2.2

We expand ϕ_t into a truncated Duhamel series

$$\phi_t = \sum_{n=0}^{N-1} \phi_{n,t} + R_{N,t}, \quad (37)$$

where

$$\phi_{n,t} := (-i\lambda)^n \int_{\mathbb{R}_+^{n+1}} ds_0 \cdots ds_n \delta\left(\sum_{j=0}^n s_j - t\right) e^{is_0 \frac{\Delta}{2}} V_\omega e^{is_1 \frac{\Delta}{2}} \cdots V_\omega e^{is_n \frac{\Delta}{2}} \phi_0 \quad (38)$$

denotes the n -th Duhamel term, and where

$$R_{N,t} = -i\lambda \int_0^t ds e^{-i(t-s)H_\omega} V_\omega \phi_{N-1,s} \quad (39)$$

is the remainder term. Here and in the sequel, we write $\phi_0 \equiv \phi_0^\eta$ for brevity. The number N remains to be optimized. Since $\|V_\omega\|_{\ell^1(\Lambda_L)} < \infty$ a.s., \widehat{V}_ω is well-defined and bounded on Λ_L^* , with probability one, for every $L < \infty$. Then,

$$\begin{aligned} \widehat{\phi}_{n,t}(k_0) &= (-i\lambda)^n \int ds_0 \cdots ds_n \delta\left(\sum_{j=0}^n s_j - t\right) \\ &\quad \int_{(\Lambda_L^*)^n} dk_1 \cdots dk_n e^{-is_0 e_\Delta(k_0)} \widehat{V}_\omega(k_1 - k_0) e^{-is_1 e_\Delta(k_1)} \cdots \\ &\quad \cdots \widehat{V}_\omega(k_n - k_{n-1}) e^{-is_n e_\Delta(k_n)} \widehat{\phi}_0(k_n) . \end{aligned} \quad (40)$$

Expressed as a resolvent expansion in momentum space, we find

$$\begin{aligned} \widehat{\phi}_{n,t}(k_0) &= \frac{(-\lambda)^n}{2\pi i} e^{\varepsilon t} \int_{\mathbb{R}} d\alpha e^{-it\alpha} \\ &\quad \int_{(\Lambda_L^*)^n} dk_1 \cdots dk_n \frac{1}{e_\Delta(k_0) - \alpha - i\varepsilon} \widehat{V}_\omega(k_1 - k_0) \\ &\quad \cdots \widehat{V}_\omega(k_n - k_{n-1}) \frac{1}{e_\Delta(k_n) - \alpha - i\varepsilon} \widehat{\phi}_0(k_n) . \end{aligned} \quad (41)$$

We refer to the Fourier multiplier $\frac{1}{e_\Delta(k) - \alpha - i\varepsilon}$ as a *particle propagator*. Likewise, we note that (41) is equivalent to the n -th term in the resolvent expansion of

$$\phi_t = \frac{1}{2\pi i} \int_{-i\varepsilon + \mathbb{R}} dz e^{-itz} \frac{1}{H_\omega - z} \phi_0 . \quad (42)$$

By the analyticity of the integrand in (41) with respect to the variable α , the path of the α -integration can, for any fixed $n \in \mathbb{N}$, be deformed into the closed contour

$$I = I_0 \cup I_1 , \quad (43)$$

away from \mathbb{R} , with

$$\begin{aligned} I_0 &:= [-1, 13] \\ I_1 &:= ([-1, 13] - i) \cup (-1 - i(0, 1]) \cup (13 - i(0, 1]) , \end{aligned}$$

which encloses $\text{spec}(-\Delta - i\varepsilon) = [0, 12] - i\varepsilon$.

Next, we apply the time partitioning method introduced in [4]. To this end, we choose $\kappa \in \mathbb{N}$ with $1 \ll \kappa \ll \varepsilon^{-1}$, and subdivide $[0, t]$ into κ subintervals bounded by $\theta_j = \frac{j t}{\kappa}$, $j = 1, \dots, \kappa$. Then,

$$R_{N,t} = -i\lambda \sum_{j=0}^{\kappa-1} e^{-i(t-\theta_{j+1})H_\omega} \int_{\theta_j}^{\theta_{j+1}} ds e^{-i(\theta_{j+1}-s)H_\omega} V_\omega \phi_{N-1,s} . \quad (44)$$

Let $\phi_{n,N,\theta}(s)$ denote the n -th Duhamel term, conditioned on the requirement that the first N collisions occur in the time interval $[0, \theta]$, and all remaining $n - N$ collisions take place in the time interval $(\theta, s]$. That is,

$$\begin{aligned} \phi_{n,N,\theta}(s) &:= (-i\lambda)^{n-N} \int_{\mathbb{R}_+^{n-N+1}} ds_0 \cdots ds_{n-N} \delta\left(\sum_{j=0}^{n-N} s_j - (s - \theta)\right) \\ &\quad e^{is_0 \frac{\Delta}{2}} V_\omega \cdots V_\omega e^{is_{n-N} \frac{\Delta}{2}} V_\omega \phi_{N-1,\theta} . \end{aligned} \quad (45)$$

Moreover, let

$$\tilde{\phi}_{n,N,\theta}(s) := -i\lambda V_\omega \phi_{n-1,N,\theta}(s) \quad (46)$$

denote its "truncated" counterpart. Further expanding e^{-isH_ω} in (44) into a truncated Duhamel series with $3N$ terms, we find

$$R_{N,t} = R_{N,t}^{(<4N)} + R_{N,t}^{(4N)}, \quad (47)$$

where

$$R_{N,t}^{(<4N)} = \sum_{j=1}^{\kappa} \sum_{n=N}^{4N-1} e^{-i(t-\theta_j)H_\omega} \phi_{n,N,\theta_{j-1}}(\theta_j) \quad (48)$$

and

$$R_{N,t}^{(4N)} = \sum_{j=1}^{\kappa} e^{-i(t-\theta_j)H_\omega} \int_{\theta_{j-1}}^{\theta_j} ds e^{-i(\theta_j-s)H_\omega} \tilde{\phi}_{4N,N,\theta_{j-1}}(s). \quad (49)$$

By the Schwarz inequality,

$$\|R_{N,t}^{(<4N)}\|_2 \leq 3N\kappa \sup_{N \leq n < 4N, 1 \leq j \leq \kappa} \|\phi_{n,N,\theta_{j-1}}(\theta_j)\|_2 \quad (50)$$

and

$$\|R_{N,t}^{(4N)}\|_2 \leq t \sup_{1 \leq j \leq \kappa} \sup_{s \in [\theta_{j-1}, \theta_j]} \|\tilde{\phi}_{4N,N,\theta_{j-1}}(s)\|_2, \quad (51)$$

for every fixed realization of V_ω .

Let $r \in 2\mathbb{N}$, and let

$$W_{t;n_1,n_2}(x,v) := 8 \sum_{\substack{y,z \in \Lambda_L \\ y+z=2x}} \overline{\psi_{n_2,t}(y)} \psi_{n_1,t}(z) e^{2\pi i(y-z) \cdot v}, \quad (52)$$

for $x \in \Lambda_{L,\frac{1}{2}}$, denote the (n_1, n_2) -th term in the Wigner distribution, with

$$\psi_{n,t} := \begin{cases} \phi_{n,t} & \text{if } n < N \\ \sum_{j=1}^{\kappa} e^{-i(t-\theta_j)H_\omega} \phi_{n,N,\theta_{j-1}}(\theta_j) & \text{if } N \leq n < 4N \\ R_{N,t}^{(4N)} & \text{if } n = 4N. \end{cases} \quad (53)$$

We note that Fourier transformation with respect to $x \in \Lambda_{L,\frac{1}{2}}$ (see (8)) yields

$$\widehat{W}_{t;n_1,n_2}(\xi, v) = \overline{\widehat{\psi}_{n_2,t}(v - \frac{\xi}{2})} \widehat{\psi}_{n_1,t}(v + \frac{\xi}{2}), \quad (54)$$

see also (13). Then, clearly,

$$\begin{aligned} & \left(\mathbb{E} \left[\langle \widehat{J}_{\lambda^2}, \widehat{W}_{\phi_t} \rangle - \mathbb{E} \langle \widehat{J}_{\lambda^2}, \widehat{W}_{\phi_t} \rangle \right]^r \right)^{\frac{1}{r}} \\ & \leq C N \sum_{n_1, n_2=0}^{4N} \left(\mathbb{E} \left[|\langle \widehat{J}_{\lambda^2}, \widehat{W}_{t;n_1,n_2} \rangle - \mathbb{E} \langle \widehat{J}_{\lambda^2}, \widehat{W}_{t;n_1,n_2} \rangle|^r \right] \right)^{\frac{1}{r}}, \end{aligned} \quad (55)$$

and we distinguish the following cases.

If $n_1, n_2 < N$, we note that

$$\begin{aligned}
& \left(\mathbb{E} \left[\left| \langle \widehat{J}_{\lambda^2}, \widehat{W}_{t;n_1,n_2} \rangle - \mathbb{E} \langle \widehat{J}_{\lambda^2}, \widehat{W}_{t;n_1,n_2} \rangle \right|^r \right] \right)^{\frac{1}{r}} \\
&= \left(\mathbb{E}_{2\text{-conn}} \left[\left| \langle \widehat{J}_{\lambda^2}, \widehat{W}_{t;n_1,n_2} \rangle \right|^r \right] \right)^{\frac{1}{r}} \\
&= \left(\mathbb{E}_{2\text{-conn}} \left[\left(\left| \int d\xi dv \widehat{J}_{\lambda^2}(\xi, v) \phi_{n_2,t}(v - \frac{\xi}{2}) \phi_{n_1,t}(v + \frac{\xi}{2}) \right|^2 \right)^{\frac{r}{2}} \right] \right)^{\frac{1}{r}}, \quad (56)
\end{aligned}$$

where $\mathbb{E}_{2\text{-conn}}$ denotes the expectation based on *2-connected graphs*, cf. Definition 5.1 below.

If $N \leq n_i \leq 4N$ for at least one value of i , we use

$$\begin{aligned}
|\langle \widehat{J}_{\lambda^2}, \widehat{W}_{t;n_1,n_2} \rangle| &= \left| \int d\xi dv \widehat{J}_{\lambda^2}(\xi, v) \overline{\psi_{n_2,t}(v - \frac{\xi}{2})} \psi_{n_1,t}(v + \frac{\xi}{2}) \right| \\
&\leq \left(\int d\xi \sup_v |\widehat{J}_{\lambda^2}(\xi, v)| \right) \|\psi_{n_1,t}\|_2 \|\psi_{n_2,t}\|_2 \quad (57)
\end{aligned}$$

and

$$\int_{2\mathbb{T}^3} d\xi \sup_v |\widehat{J}_{\lambda^2}(\xi, v)| < c. \quad (58)$$

Then, for constants C which are independent of ε , we obtain the following estimates.

If $n_1 < N$ and $N \leq n_2 < 4N$, the Schwarz inequality implies

$$\begin{aligned}
& \left(\mathbb{E} \left[\left| \langle \widehat{J}_{\lambda^2}, \widehat{W}_{t;n_1,n_2} \rangle - \mathbb{E} \langle \widehat{J}_{\lambda^2}, \widehat{W}_{t;n_1,n_2} \rangle \right|^r \right] \right)^{\frac{1}{r}} \\
&\leq C \left\{ \left(\mathbb{E} \left[\|\psi_{n_1,t}\|_2^r \|\psi_{n_2,t}\|_2^r \right] \right)^{\frac{1}{r}} + \mathbb{E} \left[\|\psi_{n_1,t}\|_2 \|\psi_{n_2,t}\|_2 \right] \right\} \quad (59) \\
&\leq C \left\{ \left(\mathbb{E} \left[\|\psi_{n_1,t}\|_2^{2r} \right] \right)^{\frac{1}{2r}} \left(\mathbb{E} \left[\|\psi_{n_2,t}\|_2^{2r} \right] \right)^{\frac{1}{2r}} + \left(\mathbb{E} \left[\|\psi_{n_1,t}\|_2^2 \right] \mathbb{E} \left[\|\psi_{n_2,t}\|_2^2 \right] \right)^{\frac{1}{2}} \right\}.
\end{aligned}$$

Thus, if $n_1 < N$, $N \leq n_2 < 4N$,

$$\begin{aligned}
& \left(\mathbb{E} \left[\left| \langle \widehat{J}_{\lambda^2}, \widehat{W}_{t;n_1,n_2} \rangle - \mathbb{E} \langle \widehat{J}_{\lambda^2}, \widehat{W}_{t;n_1,n_2} \rangle \right|^r \right] \right)^{\frac{1}{r}} \\
&\leq C \kappa N \left\{ \sup_j \left(\mathbb{E} \left[\|\phi_{n_1,t}\|_2^{2r} \right] \right)^{\frac{1}{2r}} \left(\mathbb{E} \left[\|\phi_{n_2,N,\theta_{j-1}}(\theta_j)\|_2^{2r} \right] \right)^{\frac{1}{2r}} \right. \\
&\quad \left. + \sup_j \left(\mathbb{E} \left[\|\phi_{n_1,t}\|_2^2 \right] \mathbb{E} \left[\|\phi_{n_2,N,\theta_{j-1}}(\theta_j)\|_2^2 \right] \right)^{\frac{1}{2}} \right\}. \quad (60)
\end{aligned}$$

while for $n_1 < N$, $n_2 = 4N$,

$$\begin{aligned}
& \left(\mathbb{E} \left[\left| \langle \widehat{J}_{\lambda^2}, \widehat{W}_{t;n_1,4N} \rangle - \mathbb{E} \langle \widehat{J}_{\lambda^2}, \widehat{W}_{t;n_1,4N} \rangle \right|^r \right] \right)^{\frac{1}{r}} \\
&\leq C t \left\{ \sup_j \sup_{s \in [\theta_{j-1}, \theta_j]} \left(\mathbb{E} \left[\|\phi_{n_1,t}\|_2^{2r} \right] \right)^{\frac{1}{2r}} \left(\mathbb{E} \left[\|\widetilde{\phi}_{4N,N,\theta_{j-1}}(s)\|_2^{2r} \right] \right)^{\frac{1}{2r}} \right. \\
&\quad \left. + \sup_j \sup_{s \in [\theta_{j-1}, \theta_j]} \left(\mathbb{E} \left[\|\phi_{n_1,t}\|_2^2 \right] \mathbb{E} \left[\|\widetilde{\phi}_{4N,N,\theta_{j-1}}(s)\|_2^2 \right] \right)^{\frac{1}{2}} \right\}. \quad (61)
\end{aligned}$$

If $N \leq n_1, n_2 \leq 4N$, we use the Schwarz inequality in the form

$$\begin{aligned}
& \left(\mathbb{E} \left[\left| \langle \widehat{J}_{\lambda^2}, \widehat{W}_{t;n_1,n_2} \rangle - \mathbb{E} \langle \widehat{J}_{\lambda^2}, \widehat{W}_{t;n_1,n_2} \rangle \right|^r \right] \right)^{\frac{1}{r}} \\
& \leq C \left\{ \left(\mathbb{E} \left[\left(\|\psi_{n_1,t}\|_2 + \|\psi_{n_2,t}\|_2 \right)^{2r} \right] \right)^{\frac{1}{r}} + \mathbb{E} \left[\|\psi_{n_1,t}\|_2 \|\psi_{n_2,t}\|_2 \right] \right\} \\
& \leq C \sum_{j=1}^2 \left\{ \left(\mathbb{E} \left[\|\psi_{n_j,t}\|_2^{2r} \right] \right)^{\frac{1}{r}} + \mathbb{E} \left[\|\psi_{n_j,t}\|_2^2 \right] \right\}. \tag{62}
\end{aligned}$$

Hence, for $N \leq n_1, n_2 \leq 4N$,

$$\begin{aligned}
& \sum_{N \leq n_1, n_2 \leq 4N} \left(\mathbb{E} \left[\left| \langle \widehat{J}_{\lambda^2}, \widehat{W}_{t;n_1,n_2} \rangle - \mathbb{E} \langle \widehat{J}_{\lambda^2}, \widehat{W}_{t;n_1,n_2} \rangle \right|^r \right] \right)^{\frac{1}{r}} \\
& \leq C (N\kappa)^2 \left\{ \sup_j \sup_{N \leq n < 4N} \left(\mathbb{E} \left[\|\phi_{n,N,\theta_{j-1}}(\theta_j)\|_2^{2r} \right] \right)^{\frac{1}{r}} \right. \\
& \quad \left. + \sup_j \sup_{N \leq n < 4N} \mathbb{E} \left[\|\phi_{n,N,\theta_{j-1}}(\theta_j)\|_2^2 \right] \right\} \\
& \quad + C t^2 \left\{ \sup_j \sup_{s \in [\theta_{j-1}, \theta_j]} \left(\mathbb{E} \left[\|\widetilde{\phi}_{4N,N,\theta_{j-1}}(s)\|_2^{2r} \right] \right)^{\frac{1}{r}} \right. \\
& \quad \left. + \sup_j \sup_{s \in [\theta_{j-1}, \theta_j]} \mathbb{E} \left[\|\widetilde{\phi}_{4N,N,\theta_{j-1}}(s)\|_2^2 \right] \right\}. \tag{63}
\end{aligned}$$

We shall next use Lemmata 4.1, 4.2, and 4.3 below to bound the above sums.

From Lemma 4.1, and $((nr)!)^{\frac{1}{r}} < n^n r^n$, one obtains

$$\begin{aligned}
& \sum_{n_1, n_2 < N} \left(\mathbb{E}_{2\text{-conn}} \left[\left| \langle \widehat{J}_{\lambda^2}, \widehat{W}_{t;n_1,n_2} \rangle \right|^r \right] \right)^{\frac{1}{r}} \\
& \leq C \varepsilon^{\frac{1}{5r}} N^{N+2} \left(\log \frac{1}{\varepsilon} \right)^3 (c r \lambda^2 \varepsilon^{-1} \log \frac{1}{\varepsilon})^N. \tag{64}
\end{aligned}$$

From (75) and Lemma 4.2,

$$\begin{aligned}
& \left(\sum_{\substack{n_1 < N \\ N \leq n_2 < 4N}} + \sum_{\substack{n_2 < N \\ N \leq n_1 < 4N}} \right) \left(\mathbb{E} \left[\left| \langle \widehat{J}_{\lambda^2}, \widehat{W}_{t;n_1,n_2} \rangle - \mathbb{E} \langle \widehat{J}_{\lambda^2}, \widehat{W}_{t;n_1,n_2} \rangle \right|^r \right] \right)^{\frac{1}{r}} \\
& \leq \left[C \kappa N^{N+3} \left(\log \frac{1}{\varepsilon} \right)^3 (c r \lambda^2 \varepsilon^{-1} \log \frac{1}{\varepsilon})^N \right. \\
& \quad \left. \left[\frac{(c \lambda^2 \varepsilon^{-1})^{4N}}{\sqrt{N!}} + (4N)^{4N} \left(\varepsilon^{\frac{1}{5}} + \varepsilon^{\frac{1}{5r}} \right) \left(\log \frac{1}{\varepsilon} \right)^3 (c r \lambda^2 \varepsilon^{-1} \log \frac{1}{\varepsilon})^{4N} \right] \right]^{\frac{1}{2}}. \tag{65}
\end{aligned}$$

From (75) and Lemma 4.3,

$$\begin{aligned}
& \left(\sum_{\substack{n_1 < N \\ n_2 = 4N}} + \sum_{\substack{n_2 < N \\ n_1 = 4N}} \right) \left(\mathbb{E} \left[\left| \langle \widehat{J}_{\lambda^2}, \widehat{W}_{t;n_1,4N} \rangle - \mathbb{E} \langle \widehat{J}_{\lambda^2}, \widehat{W}_{t;n_1,4N} \rangle \right|^r \right] \right)^{\frac{1}{r}} \\
& \leq \left(C \varepsilon^{-1} N^{5N+1} \kappa^{-2N} \left(\log \frac{1}{\varepsilon} \right)^6 (c r \lambda^2 \varepsilon^{-1} \log \frac{1}{\varepsilon})^{5N} \right)^{\frac{1}{2}}. \tag{66}
\end{aligned}$$

Finally, from Lemmata 4.2 and 4.3,

$$\begin{aligned}
& \sum_{N \leq n_1, n_2 \leq 4N} \left(\mathbb{E} \left[\left| \langle \widehat{J}_{\lambda^2}, \widehat{W}_{t; n_1, n_2} \rangle - \mathbb{E} \langle \widehat{J}_{\lambda^2}, \widehat{W}_{t; n_1, n_2} \rangle \right|^r \right] \right)^{\frac{1}{r}} \\
& \leq C (N\kappa)^2 \frac{(c\lambda^2 \varepsilon^{-1})^{4N}}{\sqrt{N!}} \\
& \quad + C (N\kappa)^2 (4N)^{4N} \left(\varepsilon^{\frac{1}{5}} + \varepsilon^{\frac{1}{5r}} \right) \left(\log \frac{1}{\varepsilon} \right)^3 (cr\lambda^2 \varepsilon^{-1} \log \frac{1}{\varepsilon})^{4N} \\
& \quad + C \varepsilon^{-2} \kappa^{-2N} (4N)^{4N} \left(\log \frac{1}{\varepsilon} \right)^3 (cr\lambda^2 \varepsilon^{-1} \log \frac{1}{\varepsilon})^{4N}, \tag{67}
\end{aligned}$$

We emphasize that the bounds (64) - (67) are *uniform* in L .

Consequently, for a choice of parameters

$$\begin{aligned}
\varepsilon &= \frac{1}{t} = \frac{\lambda^2}{T} \\
N &= \left\lfloor \frac{\log \frac{1}{\varepsilon}}{100 r \log \log \frac{1}{\varepsilon}} \right\rfloor \\
\kappa &= \left\lceil \left(\log \frac{1}{\varepsilon} \right)^{150r} \right\rceil, \tag{68}
\end{aligned}$$

we find, for sufficiently small ε ,

$$\begin{aligned}
\varepsilon^{-\frac{1}{70r}} &< N^N < \varepsilon^{-\frac{1}{100r}} \\
(4N)^{4N} &< \varepsilon^{-\frac{1}{20r}} \\
(cr\lambda^2 \varepsilon^{-1} \log \frac{1}{\varepsilon})^{4N} &< \varepsilon^{-\frac{1}{50r}} \\
\frac{(c\lambda^2 \varepsilon^{-1})^{4N}}{\sqrt{N!}} &< \varepsilon^{\frac{1}{25r}} \\
\kappa^{-2N} &\leq \varepsilon^3, \tag{69}
\end{aligned}$$

whereby it is easy to verify that

$$(64) < \left(\log \frac{1}{\varepsilon} \right)^{10} \varepsilon^{\frac{1}{5r}} \varepsilon^{-\frac{1}{100r}} \varepsilon^{\frac{1}{50r}} < \varepsilon^{\frac{1}{20r}}$$

$$(65) < \left(\left(\log \frac{1}{\varepsilon} \right)^{10+150r} \varepsilon^{-\frac{1}{50r}} \left(\varepsilon^{\frac{1}{25r}} + \varepsilon^{-\frac{1}{20r}} \left(\varepsilon^{\frac{1}{5}} + \varepsilon^{\frac{1}{5r}} \right) \varepsilon^{-\frac{1}{50r}} \right) \right)^{\frac{1}{2}} < \varepsilon^{\frac{1}{120r}}$$

$$(66) < \left(\left(\log \frac{1}{\varepsilon} \right)^{10} \varepsilon^{-1-\frac{1}{20r}} \varepsilon^3 \varepsilon^{-\frac{1}{50r}} \right)^{\frac{1}{2}} < \varepsilon^{\frac{1}{2}}$$

$$\begin{aligned}
(67) &< \left(\log \frac{1}{\varepsilon} \right)^{2+300r} \varepsilon^{\frac{1}{25r}} + \left(\log \frac{1}{\varepsilon} \right)^{6+300r} \varepsilon^{-\frac{1}{20r}} \left(\varepsilon^{\frac{1}{5}} + \varepsilon^{\frac{1}{5r}} \right) \varepsilon^{-\frac{1}{50r}} \\
&\quad + \left(\log \frac{1}{\varepsilon} \right)^3 \varepsilon^{-2} \varepsilon^3 \varepsilon^{-\frac{1}{20r}} \varepsilon^{-\frac{1}{50r}} < \varepsilon^{\frac{1}{30r}}. \tag{70}
\end{aligned}$$

Collecting all of the above, and recalling (55),

$$\left(\mathbb{E} \left[\left| \langle \widehat{J}_{\lambda^2}, \widehat{W}_{\phi_t} \rangle - \mathbb{E} \langle \widehat{J}_{\lambda^2}, \widehat{W}_{\phi_t} \rangle \right|^r \right] \right)^{\frac{1}{r}} < C N \varepsilon^{\frac{1}{120r}} < \varepsilon^{\frac{1}{150r}}, \tag{71}$$

uniformly in L . Hence, using Theorem 2.1,

$$\begin{aligned} & \lim_{\lambda \rightarrow 0} \lim_{L \rightarrow \infty} \left(\mathbb{E} \left[\left(\langle J, W_T^{(\lambda^2)} \rangle - \langle J, F_T \rangle \right)^r \right] \right)^{\frac{1}{r}} \\ & \leq \lim_{\lambda \rightarrow 0} \lim_{L \rightarrow \infty} \left(\mathbb{E} \left[\left(\langle \widehat{J}_{\lambda^2}, \widehat{W}_{\phi_t} \rangle - \mathbb{E} \langle \widehat{J}_{\lambda^2}, \widehat{W}_{\phi_t} \rangle \right)^r \right] \right)^{\frac{1}{r}} \\ & \quad + \lim_{\lambda \rightarrow 0} \lim_{L \rightarrow \infty} \left| \mathbb{E} \langle J, W_T^{(\lambda^2)} \rangle - \langle J, F_T \rangle \right| = 0, \end{aligned} \quad (72)$$

for every fixed, finite value of $r \in 2\mathbb{N}$ and $T > 0$. This in turn implies that (72) holds for any fixed, finite $r \in \mathbb{R}_+$ and globally in T , which establishes Theorem 2.2.

4. MAIN LEMMATA

In this section, we summarize the key technical lemmata needed to establish (71). The proofs are based on graph expansion techniques and estimation of high dimensional singular integrals in momentum space. To arrive at our results, we have to significantly generalize and extend methods developed in [4] and [2]. In all that follows, we suppose that L is finite, but much larger than any relevant scale of the problem; for our purposes, the assumption that

$$L \gg \varepsilon^{-r/\varepsilon} \quad (73)$$

will suffice.

Lemma 4.1. *Let $\bar{n} := n_1 + n_2$, where $n_1, n_2 < N$. For any fixed $r \in 2\mathbb{N}$, and every $T = \lambda^2 \varepsilon^{-1} > 0$, there exists a finite constant $c = c(T)$ independent of L such that*

$$\left(\mathbb{E}_{2\text{-conn}} \left[\left| \langle \widehat{J}_{\lambda^2}, \widehat{W}_{t;n_1,n_2} \rangle \right|^r \right] \right)^{\frac{1}{r}} \leq \varepsilon^{\frac{1}{5r}} \left(\left(\frac{\bar{n}r}{2} \right)! \right)^{\frac{1}{r}} \left(\log \frac{1}{\varepsilon} \right)^3 (c\lambda^2 \varepsilon^{-1} \log \frac{1}{\varepsilon})^{\frac{\bar{n}}{2}}. \quad (74)$$

Furthermore, for any fixed $r \in 2\mathbb{N}$ and $n < N$, there is an a priori bound

$$\left(\mathbb{E} \left[\|\phi_{n,t}\|_2^{2r} \right] \right)^{\frac{1}{r}} \leq ((nr)!)^{\frac{1}{r}} \left(\log \frac{1}{\varepsilon} \right)^3 (c\lambda^2 \varepsilon^{-1} \log \frac{1}{\varepsilon})^n, \quad (75)$$

where c is independent of T and L .

The gain of a factor $\varepsilon^{\frac{1}{5r}}$ in (74) over the a priori bound (75) is the key ingredient in our proof of (71).

Lemma 4.2. *For any fixed $r \in 2\mathbb{N}$, $N \leq n < 4N$, and $T = \lambda^2 \varepsilon^{-1}$,*

$$\begin{aligned} & \left(\mathbb{E} \left[\|\phi_{n,N,\theta_{j-1}}(\theta_j)\|_2^{2r} \right] \right)^{\frac{1}{r}} \\ & \leq \frac{(c\lambda^2 \varepsilon^{-1})^n}{\sqrt{n!}} + \left((n!) \varepsilon^{\frac{1}{5}} + ((nr)!)^{\frac{1}{r}} \varepsilon^{\frac{1}{5r}} \right) \left(\log \frac{1}{\varepsilon} \right)^3 (c'\lambda^2 \varepsilon^{-1} \log \frac{1}{\varepsilon})^n, \end{aligned} \quad (76)$$

for finite constants c and $c' = c'(T)$ which are independent of L .

This lemma is proved in Section 6.

Lemma 4.3. *For any fixed $r \in 2\mathbb{N}$ and $T > 0$, there exists a finite constant $c = c(T)$ independent of L such that*

$$\left(\mathbb{E} \left[\|\widetilde{\phi}_{4N,N,\theta_{j-1}}(\theta_j)\|_2^{2r} \right] \right)^{\frac{1}{r}} \leq \frac{((4Nr)!)^{\frac{1}{2r}} \left(\log \frac{1}{\varepsilon} \right)^3 (c\lambda^2 \varepsilon^{-1} \log \frac{1}{\varepsilon})^{4N}}{\kappa^{2N}}. \quad (77)$$

The proof of this lemma is given in Section 7.

We will make extensive use of the basic inequalities formulated in the following lemma.

Lemma 4.4. *For L sufficiently large (e.g. for (73)),*

$$\sup_{\alpha \in I, k \in \mathbb{T}^3} \frac{1}{|e_\Delta(k) - \alpha - i\varepsilon|} = \frac{1}{\varepsilon}$$

$$\int_{\mathbb{T}^3} \frac{dk}{|e_\Delta(k) - \alpha - i\varepsilon|} , \quad \int_I \frac{|d\alpha|}{|e_\Delta(k) - \alpha - i\varepsilon|} < c \log \frac{1}{\varepsilon} . \quad (78)$$

Proof. Clearly,

$$\int_{\Lambda_L^*} \frac{dk}{|e_\Delta(k) - \alpha - i\varepsilon|} \leq \int_{\mathbb{T}^3} \frac{dk}{|e_\Delta(k) - \alpha - i\varepsilon|} + O\left(\frac{1}{\varepsilon^2 |\Lambda_L|}\right) . \quad (79)$$

The bound $\int_{\mathbb{T}^3} \frac{dk}{|e_\Delta(k) - \alpha - i\varepsilon|} < c \log \frac{1}{\varepsilon}$ is proved in [2, 4]. The remaining cases are evident. \square

Moreover, we point out the following key property of the functions $\phi_{n,N,\theta_{j-1}}(s)$. From

$$\begin{aligned} \phi_{n,N,\theta}(s) &= (-i\lambda)^{n-N} \int_{\mathbb{R}_+^{n-N+1}} ds_0 \cdots ds_{n-N} \delta\left(\sum_{j=0}^{n-N} s_j - s\right) \\ &\quad e^{is_0 \frac{\Delta}{2}} V_\omega \cdots V_\omega e^{is_{n-N} \frac{\Delta}{2}} V_\omega \phi_{N-1,\theta} , \end{aligned} \quad (80)$$

where $s \in [\theta, \theta']$ and $\theta' - \theta = \frac{t}{\kappa}$, we find

$$\begin{aligned} (\widehat{\phi}_{n,N,\theta}(\theta'))(k_0) &= \frac{(-\lambda)^{n-N} e^{(s-\theta)\kappa\varepsilon}}{2\pi i} \int_I d\alpha e^{-i(s-\theta)\alpha} \int_{(\Lambda_L^*)^{n-N}} dk_1 \cdots dk_{n-N} \\ &\quad \frac{1}{e_\Delta(k_0) - \alpha - i\kappa\varepsilon} \widehat{V}_\omega(k_1 - k_0) \cdots \\ &\quad \cdots \frac{1}{e_\Delta(k_{n-N}) - \alpha - i\kappa\varepsilon} \widehat{V}_\omega(k_{n-N+1} - k_{n-N}) \\ &\quad \widehat{\phi}_{N-1,\theta}(k_{n-N+1}) , \end{aligned} \quad (81)$$

recalling that $\varepsilon = \frac{1}{t}$, and with

$$\begin{aligned} \widehat{\phi}_{N-1,\theta}(k_{n-N+1}) &= \frac{(-\lambda)^N e^{\varepsilon\theta}}{2\pi i} \int_I d\alpha e^{-i\alpha\theta} \int_{(\Lambda_L^*)^N} \prod_{j=n-N+1}^n dk_j \\ &\quad \frac{1}{e_\Delta(k_{n-N+1}) - \alpha - i\varepsilon} \widehat{V}_\omega(k_{n-N+2} - k_{n-N+1}) \cdots \\ &\quad \cdots \widehat{V}_\omega(k_n - k_{n-1}) \frac{1}{e_\Delta(k_n) - \alpha - i\varepsilon} \widehat{\phi}_0(k_n) . \end{aligned} \quad (82)$$

The key observation here is that there are $n - N + 1$ propagators with imaginary part $-i\kappa\varepsilon$ in the denominator, where $\kappa\varepsilon \gg \varepsilon$, and N propagators where the corresponding imaginary part is $i\varepsilon$. Therefore, we have a bound

$$\frac{1}{|e_\Delta(p) - \alpha - i\kappa\varepsilon|} \leq \frac{1}{\kappa\varepsilon} \ll \frac{1}{\varepsilon} \quad (83)$$

for $n - N + 1$ propagators, which is much smaller than the bound $\frac{1}{|e_\Delta(p) - \alpha - i\varepsilon|} \leq \frac{1}{\varepsilon}$. This gain of a factor $\frac{1}{\kappa}$ as compared to (78) is exploited in the time partitioning, and is applied systematically in the proof of Lemma 4.3.

5. PROOF OF LEMMA 4.1

We recall that

$$\begin{aligned} & \mathbb{E}_{2\text{-conn}} \left[\left| \langle \widehat{J}_{\lambda^2}, \widehat{W}_{t;n_1,n_2} \rangle \right|^r \right] \\ &= \mathbb{E}_{2\text{-conn}} \left[\left(\left| \int_{\Lambda_{L,\frac{1}{2}}^* \times \Lambda_L^*} dk dk' \widehat{J}_{\lambda^2}(k - k', \frac{k + k'}{2}) \right. \right. \right. \\ & \quad \left. \left. \left. \overline{\widehat{\phi}_{n_2,t}(k)} \widehat{\phi}_{n_1,t}(k') \right|^2 \right)^{\frac{r}{2}} \right], \quad (84) \end{aligned}$$

and note that \widehat{J}_{λ^2} forces $|k - k' \pmod{2\mathbb{T}^3}| < c\lambda^2$, while $|k + k' \pmod{2\mathbb{T}^3}|$ is essentially unrestricted. Next, we introduce the following multi-index notation. As n_1, n_2 will remain fixed in the proof, let for brevity $n \equiv n_1$, and $\bar{n} \equiv n_1 + n_2$. For $j = 1, \dots, r$, let

$$\begin{aligned} \underline{k}^{(j)} &:= (k_0^{(j)}, \dots, k_{\bar{n}+1}^{(j)}) \\ d\underline{k}^{(j)} &:= \prod_{\ell=0}^{\bar{n}+1} dk_\ell^{(j)} \\ d\underline{k}_{\widehat{J}_{\lambda^2}}^{(j)} &:= \prod_{\ell=0}^{\bar{n}+1} dk_\ell^{(j)} \widehat{J}_{\lambda^2}(k_n^{(j)} - k_{n+1}^{(j)}, \frac{k_n^{(j)} + k_{n+1}^{(j)}}{2}) \\ K^{(j)}[\underline{k}^{(j)}, \alpha_j, \beta_j, \varepsilon] &:= \prod_{\ell=0}^n \frac{1}{e_\Delta(k_\ell^{(j)}) - \alpha_j - i\varepsilon_j} \prod_{\ell'=n+1}^{\bar{n}+1} \frac{1}{e_\Delta(k_{\ell'}^{(j)}) - \beta_j + i\varepsilon_j} \\ U^{(j)}[\underline{k}^{(j)}] &:= \prod_{\ell=1}^n \widehat{V}_\omega(k_\ell^{(j)} - k_{\ell-1}^{(j)}) \prod_{\ell'=n+2}^{\bar{n}+1} \widehat{V}_\omega(k_{\ell'}^{(j)} - k_{\ell'-1}^{(j)}), \quad (85) \end{aligned}$$

where $\varepsilon_j := (-1)^j \varepsilon$, $\alpha_j \in I$, and $\beta_j \in \bar{I}$ (the complex conjugate of I). On the last line, we note that $\widehat{V}_\omega(k) = \widehat{V}_\omega(-k)$.

Moreover, we introduce the notation

$$\underline{\alpha} := (\alpha_1, \dots, \alpha_r) \quad , \quad d\underline{\alpha} := \prod_{j=1}^r d\alpha_j, \quad (86)$$

and likewise for $\underline{\beta}$, $\underline{\xi}$ and $d\underline{\beta}$, $d\underline{\xi}$.

Then,

$$\begin{aligned}
(84) \quad &= \frac{e^{2r\epsilon t} \lambda^{r\bar{n}}}{(2\pi)^{2r}} \int_{(I \times \bar{I})^r} d\alpha d\beta e^{-it \sum_{j=1}^r (-1)^j (\alpha_j - \beta_j)} \\
&\int_{(\mathbb{T}^3)^{(\bar{n}+2)r}} \left[\prod_{j=1}^r d\mathbf{k}_{\hat{J}_{\lambda^2}}^{(j)} \right] \mathbb{E}_{2-conn} \left[\prod_{j=1}^r U^{(j)}[\mathbf{k}^{(j)}] \right] \\
&\prod_{j=1}^r K^{(j)}[\mathbf{k}^{(j)}, \alpha_j, \beta_j, \epsilon] \widehat{\phi}_0^{(j)}(k_0^{(j)}) \overline{\widehat{\phi}_0^{(j)}(k_{\bar{n}+1}^{(j)})}, \quad (87)
\end{aligned}$$

where

$$\widehat{\phi}_0^{(j)} := \begin{cases} \widehat{\phi}_0 & \text{if } j \text{ is even} \\ \overline{\widehat{\phi}_0} & \text{if } j \text{ is odd.} \end{cases} \quad (88)$$

The expectation \mathbb{E}_{2-conn} (defined in (91) below) in (87) produces a sum of $O((\bar{n}r)!)^1$ singular integrals with complicated delta distribution insertions. We organize them by use of (*Feynman*) *graphs*, which we define next, see also Figure 1.

We consider graphs comprising r parallel, horizontal solid lines, which we refer to as *particle lines*, each containing \bar{n} vertices enumerated from the left, which account for copies of the random potential \widehat{V}_ω . Between the n -th and the $n+1$ -th \widehat{V}_ω -vertex, we a distinguished vertex is inserted to account for the contraction with \widehat{J}_{λ^2} (henceforth referred to as the " \widehat{J}_{λ^2} -vertex"). Then, the n edges on the left of the \widehat{J}_{λ^2} -vertex correspond to the propagators in $\widehat{\psi}_{n,t}$ resp. $\overline{\widehat{\psi}_{n,t}}$, while the $\bar{n} - n$ edges on the right correspond to those in $\widehat{\psi}_{\bar{n}-n,t}$ resp. $\overline{\widehat{\psi}_{\bar{n}-n,t}}$. We shall refer to those edges, labeled by the momentum variables $k_\ell^{(j)}$, as *propagator lines*.

The expectation produces a sum over all possible products of $\frac{r\bar{n}}{2}$ delta distributions, each standing for one contraction between a pair of random potentials. We connect every pair of mutually contracted random potentials with a dashed *contraction line*. We then identify the contraction type with the corresponding graph.

We remark that what is defined here as one particle line was referred to as a pair of particle lines joined by a \widehat{J}_{λ^2} -, or respectively, a δ -vertex in [2, 4]. Thus, according to the terminology of [2, 4], we would here be discussing the case of $2r$ particle lines. Due to the different emphasis in the work at hand, the convention introduced here appears to be more convenient.

We particularly distinguish the class of *completely disconnected graphs*, in which random potentials are mutually contracted only if they are located on the same particle line. Clearly, all of its members possess r connectivity components.

All other contraction types are referred to as *non-disconnected graphs*.

A particular subfamily of non-disconnected graphs, referred to as *2-connected graphs*, is defined by the property that every connectivity component has at least two particle lines. Accordingly, we may now provide the following definitions which were in part already anticipated in the preceding discussion.

Definition 5.1. *Let*

$$\mathbb{E}_{disc} \left[\prod_{j=1}^r U^{(j)}[\underline{k}^{(j)}] \right] := \prod_{j=1}^r \mathbb{E} \left[U^{(j)}[\underline{k}^{(j)}] \right] \quad (89)$$

include contractions among random potentials \widehat{V}_ω only if they lie on the same particle line. We refer to \mathbb{E}_{disc} as the expectation based on completely disconnected graphs.

We denote by

$$\mathbb{E}_{n-d} \left[\prod_{j=1}^r U^{(j)}[\underline{k}^{(j)}] \right] := \mathbb{E} \left[\prod_{j=1}^r U^{(j)}[\underline{k}^{(j)}] \right] - \mathbb{E}_{disc} \left[\prod_{j=1}^r U^{(j)}[\underline{k}^{(j)}] \right], \quad (90)$$

the expectation based on non-disconnected graphs, defined by the condition that there is at least one connectivity component comprising more than one particle line.

Moreover, we refer to

$$\mathbb{E}_{2-conn} \left[\prod_{j=1}^r U^{(j)}[\underline{k}^{(j)}] \right] := \mathbb{E} \left[\prod_{j=1}^r \left(U^{(j)}[\underline{k}^{(j)}] - \mathbb{E} \left[U^{(j)}[\underline{k}^{(j)}] \right] \right) \right] \quad (91)$$

as the expectation based on 2-connected graphs.

For $r\bar{n} \in 2\mathbb{N}$, let $\Pi_{r;\bar{n},n}^{(\widehat{J}_{\lambda^2})}$ denote the set of all graphs on $r \in \mathbb{N}$ particle lines, each containing \bar{n} \widehat{V}_ω -vertices, and each with the \widehat{J}_{λ^2} -vertex located between the n -th and $n+1$ -th \widehat{V}_ω -vertex. Then,

$$\mathbb{E} \left[\prod_{j=1}^r U^{(j)}[\underline{k}^{(j)}] \right] = \sum_{\pi \in \Pi_{r;\bar{n},n}^{(\widehat{J}_{\lambda^2})}} \delta_\pi(\underline{k}^{(1)}, \dots, \underline{k}^{(r)}), \quad (92)$$

where $\delta_\pi(\underline{k}^{(1)}, \dots, \underline{k}^{(r)})$ is defined as follows. There are $\frac{r\bar{n}}{2}$ pairing contractions between \widehat{V}_ω -vertices in π . Every (dashed) contraction line connects a random potential $\widehat{V}_\omega(k_{n_1+1}^{(j_1)} - k_{n_1}^{(j_1)})$ with a random potential $\widehat{V}_\omega(k_{n_2+1}^{(j_2)} - k_{n_2}^{(j_2)})$, for some pair of multi-indices $((j_1, n_1), (j_2, n_2))$ determined by π , for which

$$\mathbb{E} \left[\widehat{V}_\omega(k_{n_1+1}^{(j_1)} - k_{n_1}^{(j_1)}) \widehat{V}_\omega(k_{n_2+1}^{(j_2)} - k_{n_2}^{(j_2)}) \right] = \delta(k_{n_1+1}^{(j_1)} - k_{n_1}^{(j_1)} + k_{n_2+1}^{(j_2)} - k_{n_2}^{(j_2)}). \quad (93)$$

Then, $\delta_\pi(\underline{k}^{(1)}, \dots, \underline{k}^{(r)})$ is given by the product of deltas (where $\delta(0) = |\Lambda_L|$ and $\delta(k) = 0$ if $k \neq 0$) over all pairs of multi-indices $((n_1; j_1), (n_2; j_2))$ determined by π .

We refer to a graph with a single connectivity component as a *completely connected graph*. We denote the subset of $\Pi_{r;\bar{n},n}^{(\widehat{J}_{\lambda^2})}$ consisting of completely connected graphs by $\Pi_{r;\bar{n},n}^{(\widehat{J}_{\lambda^2})\text{conn}}$.

Clearly, any graph $\pi \in \Pi_{r;\bar{n},n}^{(\widehat{J}_{\lambda^2})}$ is the disjoint union of its completely connected components $\pi_j \in \Pi_{s_j;\bar{n},n}^{(\widehat{J}_{\lambda^2})\text{conn}}$ with $\sum s_j = r$. Accordingly, (84) factorizes into the corresponding Feynman amplitudes, $\text{Amp}_{\widehat{J}_{\lambda^2}}(\pi) = \prod_j \text{Amp}_{\widehat{J}_{\lambda^2}}(\pi_j)$.

We may thus restrict our attention to completely connected graphs.

Let $\pi \in \Pi_{s;\bar{n},n}^{(\hat{J}_{\lambda^2})^{conn}}$ with $s \geq 1$. Its Feynman amplitude is given by

$$\begin{aligned} \widetilde{\text{Amp}}_{\hat{J}_{\lambda^2}}(\pi) &:= \frac{\lambda^{s\bar{n}} e^{2s\epsilon t}}{(2\pi)^{2s}} \int_{(I \times \bar{I})^s} d\underline{\alpha} d\underline{\beta} e^{-it \sum_{j=1}^s (-1)^j (\alpha_j - \beta_j)} \\ &\int_{(\Lambda_L^*)^{(\bar{n}+2)s}} \left[\prod_{j=1}^s d\underline{k}^{(j)} \right] \delta_\pi(\underline{k}^{(1)}, \dots, \underline{k}^{(s)}) \\ &\int_{(\Lambda_{L, \frac{1}{2}}^*)^s} d\underline{\xi} \left[\prod_{j=1}^s \hat{J}_{\lambda^2}(\xi_j, \frac{k_n^{(j)} + k_{n+1}^{(j)}}{2}) \delta(k_n^{(j)} - k_{n+1}^{(j)} - \xi_j) \right] \\ &\prod_{j=1}^s K^{(j)}[\underline{k}^{(j)}, \alpha_j, \beta_j, \epsilon] \widehat{\phi}_0^{(j)}(k_0^{(j)}) \overline{\widehat{\phi}_0^{(j)}(k_{\bar{n}+1}^{(j)})}. \end{aligned} \quad (94)$$

Replacing $\int_{\Lambda_L^*} dk$ by $\int_{\mathbb{T}^3} dk$, and the scaled Kronecker deltas on Λ_L^* by delta distributions on \mathbb{T}^3 , we define

$$\begin{aligned} \text{Amp}_{\hat{J}_{\lambda^2}}(\pi) &:= \frac{\lambda^{s\bar{n}} e^{2s\epsilon t}}{(2\pi)^{2s}} \int_{(I \times \bar{I})^s} d\underline{\alpha} d\underline{\beta} e^{-it \sum_{j=1}^s (-1)^j (\alpha_j - \beta_j)} \\ &\int_{(\mathbb{T}^3)^{(\bar{n}+2)s}} \left[\prod_{j=1}^s d\underline{k}^{(j)} \right] \delta_\pi(\underline{k}^{(1)}, \dots, \underline{k}^{(s)}) \\ &\int_{(2\mathbb{T}^3)^s} d\underline{\xi} \left[\prod_{j=1}^s \hat{J}_{\lambda^2}(\xi_j, \frac{k_n^{(j)} + k_{n+1}^{(j)}}{2}) \delta(k_n^{(j)} - k_{n+1}^{(j)} - \xi_j) \right] \\ &\prod_{j=1}^s K^{(j)}[\underline{k}^{(j)}, \alpha_j, \beta_j, \epsilon] \widehat{\phi}_0^{(j)}(k_0^{(j)}) \overline{\widehat{\phi}_0^{(j)}(k_{\bar{n}+1}^{(j)})}, \end{aligned} \quad (95)$$

which is *independent* of L . It is obvious that $\widetilde{\text{Amp}}_{\hat{J}_{\lambda^2}}(\pi)$ is a discretization of $\text{Amp}_{\hat{J}_{\lambda^2}}(\pi)$ on a grid of lattice spacing $O(\frac{1}{L})$. The discretization error is bounded in the following lemma.

Lemma 5.1.

$$\left| \widetilde{\text{Amp}}_{\hat{J}_{\lambda^2}}(\pi) - \text{Amp}_{\hat{J}_{\lambda^2}}(\pi) \right| < \frac{C(\bar{n}, \epsilon)}{|\Lambda_L|}, \quad (96)$$

where $C(\bar{n}, \epsilon) \leq O\left(\frac{(s\bar{n})^2}{\epsilon^{s(\bar{n}+2)+1}}\right)$.

Proof. The integrand in $\text{Amp}_{\hat{J}_{\lambda^2}}(\pi)$ contains $s(\bar{n}+2)$ resolvent multipliers, each of which is bounded by $\frac{1}{\epsilon}$ in $L^\infty(\mathbb{T}^3)$, and by $\frac{1}{\epsilon^2}$ in $C^1(\mathbb{T}^3)$. It is demonstrated in our discussion below how to systematically integrate out all deltas in (95). Replacing the integral over \mathbb{T}^3 by the sum over Λ_L^* for each momentum remaining after integrating out the delta distributions in (95) (using $\underline{\xi}$ to integrate out the functions \hat{J}_{λ^2} , see (58)) yields an error of order $O(\frac{1}{|\Lambda_L|})$, multiplied with the sum of first derivatives of the integrand with respect to each momentum. That integrand is given by a product of $(\bar{n}+2)s$ resolvent multipliers; differentiation with respect to the momentum variables yields a sum in which each term can be bounded by $\epsilon^{-s(\bar{n}+2)-1}$. Moreover, this sum comprises no more than $(s(\bar{n}+2))^2$ terms (where $s \leq r$ is fixed). \square

For the truncated Duhamel series (37), we have to estimate amplitudes of the form (95) for \bar{n} up to $\bar{n} \leq 4N \leq O(\log \frac{1}{\varepsilon})$, see (68) and (48), (49) (for $\bar{n} = 4N$, there are 2^r propagators less, and the denominators of some propagators have an imaginary part $\frac{1}{\kappa\varepsilon}$ instead of $\frac{1}{\varepsilon}$, see Section 6; this only improves the bounds considered here). Thus, for $L \gg \varepsilon^{-r/\varepsilon} \geq \varepsilon^{-s/\varepsilon}$ (see (73)), the discretization error is smaller than $O(\varepsilon)$. Accordingly, we shall henceforth only consider $\text{Amp}_{\hat{J}_{\lambda^2}}(\pi)$, and assume L to be sufficiently large for the discretization errors to be negligible in all cases under consideration. In particular, all bounds obtained in the sequel will be *uniform in L* , and we recall that we are sending L to ∞ first before taking any other limits.

The following key lemma is in part a joint result with Laszlo Erdős.

Lemma 5.2. *Let $s \geq 2$, $s\bar{n} \in 2\mathbb{N}$, and let $\pi \in \Pi_{s;\bar{n},n}^{(\hat{J}_{\lambda^2})\text{conn}}$ be a completely connected graph. Then, there exists a finite constant $c = c(T)$ independent of L such that*

$$|\text{Amp}_{\hat{J}_{\lambda^2}}(\pi)| \leq \varepsilon^{\frac{1}{5}} (\log \frac{1}{\varepsilon})^3 (c\lambda^2 \varepsilon^{-1} \log \frac{1}{\varepsilon})^{\frac{s\bar{n}}{2}}, \quad (97)$$

for every $T = \lambda^2 \varepsilon^{-1} > 0$.

5.1. Classification of contractions. For the proof of Lemma 5.2, we classify the contractions among random potentials appearing in δ_π beyond the typification introduced in [4] and [2].

We define the following types of delta distributions.

Definition 5.2. *A delta distribution of the form*

$$\delta(k_{i+1}^{(j)} - k_i^{(j)} + k_{i'+1}^{(j)} - k_{i'}^{(j)}) \quad , \quad |i - i'| \geq 1 \quad (98)$$

which connects the i -th with the i' -th vertex on the same particle line is called an internal delta. The corresponding contraction line in the graph is an internal contraction.

An internal delta with $|i - i'| = 1$ is called an immediate recollision.

A delta distribution of the form

$$\delta(k_{i+1}^{(j)} - k_i^{(j)} + k_{i'+1}^{(j')} - k_{i'}^{(j')}) \quad , \quad j \neq j' \quad (99)$$

which connects the i -th vertex on the j -th particle line with the i' -th vertex on the j' -th particle line is called a transfer delta. The corresponding contraction line is referred to as a transfer contraction, and labeled by $((i; j), (i'; j'))$. A vertex that is adjacent to a transfer contraction is called a transfer vertex.

5.2. Reduction to the L^4 -problem. Assume that $s \geq 2$. Given a completely connected graph $\pi \in \Pi_{s;\bar{n},n}^{(\hat{J}_{\lambda^2})\text{conn}}$, we enumerate the transfer contraction lines by $\ell \in \{1, \dots, m\}$ (with m denoting the number of transfer contraction lines in π).

We decompose $\pi \in \Pi_{s;\bar{n},n}^{(\hat{J}_{\lambda^2})\text{conn}}$ into s reduced 1-particle lines as follows, see also Figure 2.

Assume that the ℓ -th transfer contraction is labeled by $((i_\ell; j), (i'_\ell; j'))$. We replace the corresponding transfer delta by the product

$$\begin{aligned} \delta(k_{i_\ell+1}^{(j)} - k_{i_\ell}^{(j)} + k_{i'_\ell+1}^{(j')} - k_{i'_\ell}^{(j')}) \\ \rightarrow \delta(k_{i_\ell+1}^{(j)} - k_{i_\ell}^{(j)} + u_\ell) \delta(k_{i'_\ell+1}^{(j')} - k_{i'_\ell}^{(j')} - u_\ell), \end{aligned} \quad (100)$$

where the first factor is attributed to the j -th, and the second factor to the j' -th particle line. We say that $\delta(k_{i_\ell+1}^{(j)} - k_{i_\ell}^{(j)} + u_\ell)$ couples the vertex $(i; j)$ to the new variable u_ℓ . We refer to u_ℓ as the *transfer momentum* corresponding to the ℓ -th transfer contraction line, and to $\delta(k_{i_\ell+1}^{(j)} - k_{i_\ell}^{(j)} + u_\ell)$ as the *reduced transfer delta* on the j -th particle line (parametrized by u_ℓ). We factorize every transfer delta, and associate each reduced transfer delta to the corresponding contraction line.

Let $\underline{u}^{(j)}$ comprise all transfer momenta u_ℓ which couple to a transfer vertex on the j -th particle line. We define

$$\delta_{int}(\underline{k}^{(j)}) := \prod_{\text{internal deltas}} \delta(k_{i+1}^{(j)} - k_i^{(j)} + k_{i'+1}^{(j)} - k_{i'}^{(j)}) \quad (101)$$

and

$$\delta^{(j)}(\underline{u}^{(j)}, \underline{k}^{(j)}) := \delta_{int}(\underline{k}^{(j)}) \prod_{\substack{u_\ell \text{ belonging to } \underline{u}^{(j)} \\ \text{on } j\text{-th particle line}}} \delta(k_{i_\ell+1}^{(j)} - k_{i_\ell}^{(j)} \pm u_\ell), \quad (102)$$

which comprises all deltas on the j -th particle line, including the corresponding factors from the modified transfer deltas.

Moreover, every vertex carries a factor λ .

Definition 5.3. *The j -th reduced 1-particle graph $\pi_j(\underline{u}^{(j)})$ comprises the j -th particle line, \bar{n} V_ω -vertices, one \widehat{J}_{λ^2} -vertex, all internal contractions, but none of the transfer contraction lines. The transfer vertices carry the reduced transfer deltas, and are parametrized by $\underline{u}^{(j)}$.*

Accordingly, we refer to

$$\begin{aligned} \text{Amp}_{\widehat{J}_{\lambda^2}}(\pi_j(\underline{u}^{(j)})) &:= \frac{\lambda^{\bar{n}} e^{2s\epsilon t}}{(2\pi)^2} \int_{I \times \bar{I}} d\alpha_j d\beta_j e^{-it(-1)^j(\alpha_j - \beta_j)} \\ &\int_{(\mathbb{T}^3)^{\bar{n}+2}} d\underline{k}^{(j)} \delta^{(j)}(\underline{u}^{(j)}, \underline{k}^{(j)}) \\ &\int_{2\mathbb{T}^3} d\xi_j \widehat{J}_{\lambda^2}(\xi_j, \frac{k_n^{(j)} + k_{n+1}^{(j)}}{2}) \delta(k_n^{(j)} - k_{n+1}^{(j)} - \xi_j) \\ &K^{(j)}[\underline{k}^{(j)}, \alpha_j, \beta_j, \epsilon] \widehat{\phi}_0^{(j)}(k_0^{(j)}) \overline{\widehat{\phi}_0^{(j)}(k_{\bar{n}+1}^{(j)})} \end{aligned} \quad (103)$$

as the j -th reduced 1-particle amplitude.

The amplitude $\text{Amp}_{\widehat{J}_{\lambda^2}}(\pi)$ is obtained from the product of all reduced 1-particle amplitudes, by integrating over the transfer momenta.

Lemma 5.3. (*Factorization lemma*) *Assume that $\pi \in \Pi_{s; \bar{n}, n}^{(\widehat{J}_{\lambda^2}) \text{ conn}}$, for $s \geq 2$, carries the transfer momenta $\underline{u} = (u_1, \dots, u_m)$. Let $\pi_j(\underline{u}^{(j)})$, for $j = 1, \dots, s$, denote the*

j -th reduced 1-particle graph. Then,

$$\text{Amp}_{\hat{J}_{\lambda^2}}(\pi) = \int du_1 \dots du_m \prod_{\ell=1}^s \text{Amp}_{\hat{J}_{\lambda^2}}(\pi_j(\underline{u}^{(j)})) . \quad (104)$$

Notably, every u_ℓ in \underline{u} appears in precisely two different reduced 1-particle amplitudes (once with each sign).

Next, we reduce the problem for $s \geq 2$ to the problem $s = 2$ (corresponding to a completely connected L^4 -graph).

To this end, let us assume that π contains m transfer contractions, carrying the transfer momenta $\underline{u} = (u_1, \dots, u_m)$. Then, by (5.3),

$$|\text{Amp}_{\hat{J}_{\lambda^2}}(\pi)| \leq \int d\underline{u} \left[\prod_{j=1}^s |\text{Amp}_{\hat{J}_{\lambda^2}}(\pi_j(\underline{u}^{(j)}))| \right] , \quad (105)$$

where $\underline{u}^{(j)}$ denotes the subset of m_j transfer momenta which couple to the j -th particle line. Moreover, let $\underline{u}^{(j;i)}$ denote the subset of transfer momenta in $\underline{u}^{(j)}$ belonging to transfer contractions between the j -th and the i -th reduced 1-particle line. We recall that every transfer momentum appears in precisely two reduced 1-particle amplitudes. Hence,

$$\underline{u}^{(j;i)} = \underline{u}^{(i;j)} \text{ for all } i \neq j, \text{ and } \underline{u}^{(i;i)} = \emptyset \text{ for all } i . \quad (106)$$

Assuming that $\underline{u}^{(s-1;s)} \neq \emptyset$ (possibly after relabeling the particle lines),

$$\begin{aligned} |\text{Amp}_{\hat{J}_{\lambda^2}}(\pi)| &\leq \int d\underline{u}^{(1;2)} \dots d\underline{u}^{(1;s)} d\underline{u}^{(2;3)} \dots \dots d\underline{u}^{(s-2;s-1)} d\underline{u}^{(s-2;s)} d\underline{u}^{(s-1;s)} \\ &\quad \left[\prod_{j=1}^s |\text{Amp}_{\hat{J}_{\lambda^2}}(\pi_j(\underline{u}^{(j;1)}, \dots, \underline{u}^{(j;j-1)}, \underline{u}^{(j;j+1)}, \dots, \underline{u}^{(j;s)}))| \right] \\ &\leq \left[\int d\underline{u}^{(1;2)} \dots d\underline{u}^{(1;s)} |\text{Amp}_{\hat{J}_{\lambda^2}}(\pi_1(\underline{u}^{(1;2)}, \dots, \underline{u}^{(1;s)}))| \right] \\ &\quad \sup_{\underline{u}^{(1;2)}} \left[\int d\underline{u}^{(2;3)} \dots d\underline{u}^{(2;s)} |\text{Amp}_{\hat{J}_{\lambda^2}}(\pi_2(\underline{u}^{(2;1)}, \underline{u}^{(2;3)}, \dots, \underline{u}^{(2;s)}))| \right] \\ &\quad \dots \dots \dots \\ &\quad \sup_{\substack{\underline{u}^{(\ell;s-1)}, \underline{u}^{(\ell;s)} \\ 1 \leq \ell \leq s-2}} \left[\int d\underline{u}^{(s-1;s)} |\text{Amp}_{\hat{J}_{\lambda^2}}(\pi_{s-1}(\underline{u}^{(s-1;1)}, \dots, \underline{u}^{(s-1;s)}))| \right. \\ &\quad \left. |\text{Amp}_{\hat{J}_{\lambda^2}}(\pi_s(\underline{u}^{(s;1)}, \dots, \underline{u}^{(s;s-1)}))| \right] \\ &= \left[\prod_{j=1}^{s-2} A_j \right] B , \end{aligned} \quad (107)$$

where

$$A_j := \sup_{\substack{\underline{u}^{(i;j)} \\ 1 \leq i < j}} \left[\int d\underline{u}^{(j;j+1)} \dots d\underline{u}^{(j;s)} |\text{Amp}_{\hat{J}_{\lambda^2}}(\pi_j(\underline{u}^{(j;1)}, \dots, \underline{u}^{(j;j-1)}, \underline{u}^{(j;j+1)}, \dots, \underline{u}^{(j;s)}))| \right] \quad (108)$$

for $1 \leq j \leq s-1$, and

$$A_s := \sup_{\substack{\underline{u}^{(i;s)} \\ 1 \leq i < s}} |\text{Amp}_{\hat{J}_{\lambda^2}}(\pi_s(\underline{u}^{(s;1)}, \dots, \underline{u}^{(s;s-1)}))| \quad (109)$$

(A_{s-1} and A_s are used in the a priori bound of Lemma 5.5 below). Moreover,

$$B := \sup_{\substack{\underline{u}^{(\ell;s-1)}, \underline{u}^{(\ell;s)} \\ 1 \leq \ell \leq s-2}} \left[\int d\underline{u}^{(s-1;s)} |\text{Amp}_{\hat{J}_{\lambda^2}}(\pi_{s-1}(\underline{u}^{(s-1;1)}, \dots, \underline{u}^{(s-1;s-1)}))| \right. \\ \left. |\text{Amp}_{\hat{J}_{\lambda^2}}(\pi_s(\underline{u}^{(s;1)}, \dots, \underline{u}^{(s;s-1)}))| \right] \quad (110)$$

corresponds to a completely connected L^4 -graph. We note that

$$B \leq A_{s-1} A_s \quad (111)$$

is evident.

Next, we estimate the terms A_j .

Lemma 5.4. *Assume that the j -th truncated particle line contains m_j transfer deltas, carrying the transfer momenta $\underline{u}^{(j)}$. Let $\underline{u}^{(j)} = (u_1^{(j)}, \dots, u_{m_j}^{(j)})$, according to an arbitrary enumeration of the transfer vertices.*

Let $a \in \mathbb{N}_0$ and $0 \leq a \leq m_j$, and arbitrarily partition $\underline{u}^{(j)}$ into $\underline{u}_1^{(j)}$ and $\underline{u}_\infty^{(j)}$, where $\underline{u}_1^{(j)}$ contains a , and $\underline{u}_\infty^{(j)}$ contains $m_j - a$ transfer momenta. Then,

$$\sup_{\underline{u}_\infty^{(j)}} \int d\underline{u}_1^{(j)} |\text{Amp}_{\hat{J}_{\lambda^2}}(\pi_j(\underline{u}^{(j)}))| < (c\lambda)^{\bar{n}} \varepsilon^{-\frac{\bar{n}+m_j}{2}+a} (\log \frac{1}{\varepsilon})^{\frac{\bar{n}-m_j}{2}+a+2}, \quad (112)$$

for a constant c which is independent of ε .

Proof. For notational convenience, we may, without any loss of generality, assume that

$$\underline{u}_1^{(j)} := (u_1^{(j)}, \dots, u_a^{(j)}), \quad \underline{u}_\infty^{(j)} := (u_{a+1}^{(j)}, \dots, u_{m_j}^{(j)}) \quad (113)$$

(by possibly relabeling the transfer momenta in $\underline{u}^{(j)}$). We recall the definition of $\text{Amp}(\pi_j(\underline{u}^{(j)}))$ from (103), and note that $\pi_j(\underline{u}^{(j)})$ contains m_j vertices carrying reduced transfer deltas, $\bar{n} - m_j \in 2\mathbb{N}$ vertices that are adjacent to an internal contraction line, and one \hat{J}_{λ^2} -vertex. Clearly,

$$\int d\underline{u}_1^{(j)} |\text{Amp}_{\hat{J}_{\lambda^2}}(\pi_j(\underline{u}^{(j)}))| \leq \frac{\lambda^{\bar{n}}}{(2\pi)^2} e^{2\varepsilon t} \left(\int_{2\mathbb{T}^3} d\xi \sup_v |\hat{J}_{\lambda^2}(\xi, v)| \right) \\ \sup_{\xi} \int_{I \times \bar{I}} |d\alpha_j| |d\beta_j| \int_{(\mathbb{T}^3)^{\bar{n}+2+a}} d\underline{u}_1^{(j)} d\underline{k}^{(j)} \delta^{(j)}(\underline{u}^{(j)}, \underline{k}^{(j)}) \delta(k_n^{(j)} - k_{n+1}^{(j)} - \xi) \\ |K^{(j)}[\underline{k}^{(j)}, \alpha_j, \beta_j, \varepsilon]| |\hat{\phi}_0^{(j)}(k_0^{(j)})| |\hat{\phi}_0^{(j)}(k_{\bar{n}+1}^{(j)})|, \quad (114)$$

and we recall (58). Adding the arguments of all delta distributions, we find the momentum conservation condition

$$k_{\bar{n}+1}^{(j)} = k_0^{(j)} + \xi + \sum_{i=1}^{m_j} (\pm u_i^{(j)}), \quad (115)$$

linking the momenta at both ends of the reduced 1-particle graph. We replace the delta belonging to the vertex $(\bar{n}; j)$ by $\delta(k_{\bar{n}+1}^{(j)} - k_0^{(j)} - \xi - \sum_{i=1}^{m_j} (\pm u_i^{(j)}))$, irrespective

of it being an internal or a reduced transfer delta, and remove it from $\delta^{(j)}(\underline{u}^{(j)}, \underline{k}^{(j)})$. We integrate out the \widehat{J}_{λ^2} -delta $\delta(k_n^{(j)} - k_{n+1}^{(j)} - \xi)$ using the variable $k_{n+1}^{(j)}$, and the delta $\delta(k_{\bar{n}+1}^{(j)} - k_0^{(j)} - \xi - \sum_{i=1}^{m_j} (\pm u_i^{(j)}))$ using the variable $k_{\bar{n}+1}^{(j)}$. It follows that if $1 \leq n < \bar{n}$,

$$|(114)| \leq C \lambda^{\bar{n}} \sup_{\xi} \int_{(\mathbb{T}^3)^a} d\underline{u}_1^{(j)} \int dk_0^{(j)} |\widehat{\phi}_0^{(j)}(k_0^{(j)})| |\widehat{\phi}_0^{(j)}(k_0^{(j)} + \xi + \sum_{i=1}^{m_j} (\pm u_i^{(j)}))|$$

$$\int_{I \times \bar{I}} |d\alpha_j| |d\beta_j| F[\underline{u}^{(j)}, k_0^{(j)}, \xi, \alpha_j, \beta_j, \varepsilon]$$

$$\frac{1}{|e_{\Delta}(k_0^{(j)}) - \alpha_j - i\varepsilon| |e_{\Delta}(k_0^{(j)} + \xi + \sum_{i=1}^{m_j} (\pm u_i^{(j)})) - \beta_j + i\varepsilon|}$$

where

$$F[\underline{u}^{(j)}, k_0^{(j)}, \xi, \alpha_j, \beta_j, \varepsilon] := \int_{(\mathbb{T}^3)^{\bar{n}-1}} dk_n^{(j)} d\underline{k}^{(j)} \widetilde{\delta}^{(j)}(\underline{u}^{(j)}, \underline{k}^{(j)}, \xi)$$

$$\frac{|\widetilde{K}^{(j)}[\underline{k}^{(j)}, \alpha_j, \beta_j, \varepsilon]|}{|e_{\Delta}(k_n^{(j)}) - \alpha_j - i\varepsilon| |e_{\Delta}(k_n^{(j)} + \xi) - \beta_j + i\varepsilon|} \quad (116)$$

with

$$\widehat{\underline{k}}^{(j)} := (k_0^{(j)}, k_n^{(j)}, \widetilde{\underline{k}}^{(j)}) \quad , \quad \widetilde{\underline{k}}^{(j)} := (k_1^{(j)}, \dots, k_{n-1}^{(j)}, k_{n+2}^{(j)}, \dots, k_{\bar{n}}^{(j)}) \quad (117)$$

and

$$\widetilde{K}^{(j)}[\underline{k}^{(j)}, \alpha_j, \beta_j, \varepsilon] := \prod_{\ell=1}^{n-1} \frac{1}{e_{\Delta}(k_{\ell}^{(j)}) - \alpha_j - i\varepsilon_j}$$

$$\prod_{\ell'=n+2}^{\bar{n}} \frac{1}{e_{\Delta}(k_{\ell'}^{(j)}) - \beta_j + i\varepsilon_j} . \quad (118)$$

Here, $\widetilde{\delta}^{(j)}(\underline{u}^{(j)}, \underline{k}^{(j)}, \xi)$ is obtained from $\delta^{(j)}(\underline{u}^{(j)}, \underline{k}^{(j)})$ by omitting the delta distribution belonging to the vertex (\bar{n}, j) , and by substituting $k_{n+1}^{(j)} \rightarrow k_n^{(j)} - \xi$. Splitting

$$|\widehat{\phi}_0^{(j)}(k_0^{(j)})| |\widehat{\phi}_0^{(j)}(k_0^{(j)} + u)| \leq \frac{1}{2} |\widehat{\phi}_0^{(j)}(k_0^{(j)})|^2 + \frac{1}{2} |\widehat{\phi}_0^{(j)}(k_0^{(j)} + u)|^2 , \quad (119)$$

we find

$$|(114)| \leq (I) + (II) , \quad (120)$$

where

$$(I) \leq C \lambda^{\bar{n}} \left[\int dk_0^{(j)} |\widehat{\phi}_0^{(j)}(k_0^{(j)})|^2 \right] \left[\sup_{k_0^{(j)}} \int_I |d\alpha_j| \frac{1}{|e_{\Delta}(k_0^{(j)}) - \alpha_j - i\varepsilon|} \right]$$

$$\sup_{\xi} \sup_{\alpha_j} \sup_{k_0^{(j)}} \int_{\bar{I}} |d\beta_j| \int_{(\mathbb{T}^3)^a} d\underline{u}_1^{(j)} \frac{F[\underline{u}^{(j)}, k_0^{(j)}, \xi, \alpha_j, \beta_j, \varepsilon]}{|e_{\Delta}(k_0^{(j)} + \xi + \sum_{i=1}^{m_j} (\pm u_i^{(j)})) - \beta_j + i\varepsilon|}$$

$$\leq C \lambda^{\bar{n}} \|\widehat{\phi}_0\|_2^2 \log \frac{1}{\varepsilon} \quad (121)$$

$$\sup_{\xi} \sup_{\alpha_j} \sup_{k_0^{(j)}} \int_{\bar{I}} |d\beta_j| \int_{(\mathbb{T}^3)^a} d\underline{u}_1^{(j)} \frac{F[\underline{u}^{(j)}, k_0^{(j)}, \xi, \alpha_j, \beta_j, \varepsilon]}{|e_{\Delta}(k_0^{(j)} + \xi + \sum_{i=1}^{m_j} (\pm u_i^{(j)})) - \beta_j + i\varepsilon|} .$$

and

$$\begin{aligned}
(II) &\leq C \lambda^{\bar{n}} \sup_{\xi} \int_{(\mathbb{T}^3)^a} d\underline{u}_1^{(j)} \int dk_0^{(j)} |\widehat{\phi}_0^{(j)}(k_0^{(j)} + \xi + \sum_{i=1}^{m_j} (\pm u_i^{(j)}))|^2 \\
&\quad \int_{I \times \bar{I}} |d\alpha_j| |d\beta_j| \frac{1}{|e_{\Delta}(k_0^{(j)}) - \alpha_j - i\varepsilon|} \\
&\quad \frac{F[\underline{u}^{(j)}, k_0^{(j)}, \xi, \alpha_j, \beta_j, \varepsilon]}{|e_{\Delta}(k_0^{(j)} + \xi + \sum_{i=1}^{m_j} (\pm u_i^{(j)})) - \beta_j + i\varepsilon|} \\
&\leq C \lambda^{\bar{n}} \left[\int dk_0^{(j)} |\widehat{\phi}_0^{(j)}(k_0^{(j)})|^2 \right] \left[\sup_{k_0^{(j)}} \int_I |d\beta_j| \frac{1}{|e_{\Delta}(k_0^{(j)}) - \beta_j - i\varepsilon|} \right] \\
&\quad \sup_{\xi} \sup_{\beta_j} \sup_{k_0^{(j)}} \int_I |d\alpha_j| \int_{(\mathbb{T}^3)^a} d\underline{u}_1^{(j)} \frac{F'[\underline{u}^{(j)}, k_0^{(j)}, \xi, \alpha_j, \beta_j, \varepsilon]}{|e_{\Delta}(k_0^{(j)} - \xi - \sum_{i=1}^{m_j} (\pm u_i^{(j)})) - \alpha_j + i\varepsilon|} \\
&\leq C \lambda^{\bar{n}} \|\widehat{\phi}_0\|_2^2 \log \frac{1}{\varepsilon} \tag{122} \\
&\quad \sup_{\xi} \sup_{\beta_j} \sup_{k_0^{(j)}} \int_I |d\alpha_j| \int_{(\mathbb{T}^3)^a} d\underline{u}_1^{(j)} \frac{F'[\underline{u}^{(j)}, k_0^{(j)}, \xi, \alpha_j, \beta_j, \varepsilon]}{|e_{\Delta}(k_0^{(j)} - \xi - \sum_{i=1}^{m_j} (\pm u_i^{(j)})) - \alpha_j + i\varepsilon|}.
\end{aligned}$$

We have here applied a shift $k_0^{(j)} \rightarrow k_0^{(j)} - \xi - \sum_{i=1}^{m_j} (\pm u_i^{(j)})$ which induces $F \rightarrow F'$ in the obvious way. We note that this only affects the delta distributions belonging to the vertices $(1, j)$ and (\bar{n}, j) in $\widetilde{\delta}^{(j)}(\underline{u}^{(j)}, \widehat{\underline{k}}^{(j)}, \xi)$ of F .

We focus on (I) , the case of (II) is analogous. We have

$$\begin{aligned}
&\sup_{\xi} \sup_{\alpha_j} \sup_{k_0^{(j)}} \int_I |d\beta_j| \int_{(\mathbb{T}^3)^a} d\underline{u}_1^{(j)} \frac{F[\underline{u}^{(j)}, k_0^{(j)}, \xi, \alpha_j, \beta_j, \varepsilon]}{|e_{\Delta}(k_0^{(j)} + \xi + \sum_{i=1}^{m_j} (\pm u_i^{(j)})) - \beta_j + i\varepsilon|} \tag{123} \\
&= \sup_{\xi} \sup_{\alpha_j} \sup_{k_0^{(j)}} \int_I |d\beta_j| \int d\underline{u}_1^{(j)} dk_n^{(j)} d\widehat{\underline{k}}^{(j)} \frac{|\widetilde{K}^{(j)}[\widehat{\underline{k}}^{(j)}, \alpha_j, \beta_j, \varepsilon]|}{|e_{\Delta}(k_0^{(j)} + \xi + \sum_{i=1}^{m_j} (\pm u_i^{(j)})) - \beta_j + i\varepsilon|} \\
&\quad \frac{\widetilde{\delta}^{(j)}(\underline{u}^{(j)}, \widehat{\underline{k}}^{(j)}, \xi)}{|e_{\Delta}(k_n^{(j)}) - \alpha_j - i\varepsilon| |e_{\Delta}(k_n^{(j)} + \xi) - \beta_j + i\varepsilon|} \\
&\leq \left[\sup_{k_n^{(j)}} \int_I |d\beta_j| \frac{1}{|e_{\Delta}(k_n^{(j)} + \xi) - \beta_j + i\varepsilon|} \right] \\
&\quad \sup_{\xi} \sup_{\alpha_j} \sup_{k_0^{(j)}} \int d\underline{u}_1^{(j)} dk_n^{(j)} d\widehat{\underline{k}}^{(j)} \frac{|\widetilde{K}^{(j)}[\widehat{\underline{k}}^{(j)}, \alpha_j, \beta_j, \varepsilon]|}{|e_{\Delta}(k_0^{(j)} + \xi + \sum_{i=1}^{m_j} (\pm u_i^{(j)})) - \beta_j + i\varepsilon|} \\
&\quad \frac{\widetilde{\delta}^{(j)}(\underline{u}^{(j)}, \widehat{\underline{k}}^{(j)}, \xi)}{|e_{\Delta}(k_n^{(j)}) - \alpha_j - i\varepsilon|}. \tag{124}
\end{aligned}$$

Next, we integrate out the reduced transfer deltas:

- The case $1 \leq \ell \leq a$: If $i_{\ell} < \bar{n}$, we integrate out the corresponding transfer deltas $\delta(k_{i_{\ell}+1}^{(j)} - k_{i_{\ell}}^{(j)} \pm u_{\ell}^{(j)})$ using the transfer momenta $u_{\ell}^{(j)}$ (the components of $\underline{u}_1^{(j)}$). Then, for each such ℓ , we use the variable $k_{i_{\ell}+1}^{(j)}$ (on the right of

the corresponding transfer vertex, according to our conventions) to estimate the corresponding propagator in L^1 ,

$$\int \frac{dk_{i_\ell+1}^{(j)}}{|e(k_{i_\ell+1}^{(j)}) - \gamma \pm i\varepsilon|} < c \log \frac{1}{\varepsilon} \quad (125)$$

(where γ denotes α_j or β_j). If the \bar{n} -th vertex is a transfer vertex, and $i_\ell = \bar{n}$, we recall that the corresponding transfer delta has already been integrated out using the momentum $k_{\bar{n}+1}^{(j)}$, and replaced by the delta enforcing (115). Accordingly, we use u_ℓ for the estimate

$$\int \frac{du_\ell}{|e_\Delta(k_0^{(j)} + \xi + \sum_{i=1}^{m_j} (\pm u_i^{(j)})) - \beta_j + i\varepsilon|} \leq c \log \frac{1}{\varepsilon}, \quad (126)$$

noting that the propagator in the integrand (supported on the edge initially labeled by $k_{\bar{n}+1}^{(j)}$) is the only one depending on u_ℓ . Thus, in this step, a propagators are in total estimated in L^1 by $c \log \frac{1}{\varepsilon}$, irrespectively whether there is $\ell \leq a$ with $i_\ell = \bar{n}$ or not.

- The case $a < \ell \leq m_j$: If $i_\ell < \bar{n}$, we integrate out the corresponding reduced transfer deltas $\delta(k_{i_\ell+1}^{(j)} - k_{i_\ell}^{(j)} \pm u_\ell^{(j)})$ using the variable $k_{i_\ell+1}^{(j)}$ on the right of the associated vertex $(i_\ell; j)$. We then estimate each of the corresponding propagators by

$$\sup_{k_{i_\ell+1}^{(j)}} \frac{1}{|e(k_{i_\ell+1}^{(j)}) - \gamma \pm i\varepsilon|} \leq \frac{1}{\varepsilon} \quad (127)$$

in L^∞ . If $i_\ell = \bar{n}$, we again note that the corresponding transfer delta has already been integrated out using the momentum $k_{\bar{n}+1}^{(j)}$. For the propagator supported on the edge labeled by $k_{\bar{n}+1}^{(j)}$, we use

$$\frac{1}{|e_\Delta(k_0^{(j)} + \xi + \sum_{i=1}^{m_j} (\pm u_i^{(j)})) - \beta_j + i\varepsilon|} \leq \frac{1}{\varepsilon}. \quad (128)$$

Thus, in this step, $m_j - a$ propagators are in total estimated in L^∞ by $\frac{1}{\varepsilon}$, irrespectively of whether there is $\ell > a$ with $i_\ell = \bar{n}$ or not.

We summarize that out of the $\bar{n} + 2$ momenta in $\underline{k}^{(j)}$, we have used $k_0^{(j)}$, $k_{\bar{n}+1}^{(j)}$, and $k_{\bar{n}+1}^{(j)}$ to begin with. Moreover, if the \bar{n} -th vertex is a transfer vertex, we have used another $m_j - 1$ components of $\underline{k}^{(j)}$ to either integrate out transfer deltas, or to estimate propagators in L^1 . On the other hand, if the \bar{n} -th vertex is an internal vertex, we have, to this end, used m_j components of $\underline{k}^{(j)}$.

We also note that out of the $\bar{n} + 2$ propagators, a have been estimated by $\frac{1}{\varepsilon}$ in L^∞ , and $m_j - a + 2$ (two from the integrals in α_j and β_j) by $c \log \frac{1}{\varepsilon}$ in L^1 .

Next, we introduce a spanning tree T on $\pi_j(\underline{u}^{(j)})$, which contains all internal contraction lines, but none of the transfer vertices, and none of the $m_j - a + 2$ edges carrying propagators that were already estimated above in L^1 or L^∞ . Thus, in particular, T does not contain the propagator edges corresponding to the momenta $k_0^{(j)}$, $k_{\bar{n}+1}^{(j)}$ and $k_{\bar{n}+1}^{(j)}$. We then call T *admissible*.

Thus, we distinguish the following cases:

- The \bar{n} -th vertex is an internal vertex: The corresponding internal delta has already been replaced by the delta enforcing (115), and integrated out using $k_{\bar{n}+1}^{(j)}$. Accordingly, we use the estimate (128) for the propagator on its right. Out of the remaining $\bar{n} - 2 - m_j$ momenta in $\underline{k}^{(j)}$, we use $\frac{\bar{n}-m_j}{2} - 1$ momenta supported on T to integrate out the remaining internal deltas, and we estimate the corresponding propagators in L^∞ by $\frac{1}{\varepsilon}$. There remain $\frac{\bar{n}-m_j}{2} - 1$ momenta for L^1 -bounds on the corresponding propagators.
- The \bar{n} -th vertex is a transfer vertex: Out of the remaining $\bar{n} - 3 - m_j$ momenta in $\underline{k}^{(j)}$, we use $\frac{\bar{n}-m_j}{2}$ momenta supported on T to integrate out the internal deltas, and we estimate the corresponding propagators in L^∞ by $\frac{1}{\varepsilon}$. There remain $\frac{\bar{n}-m_j}{2} - 2$ momenta for L^1 -bounds on the corresponding propagators.

With the rôles of the propagators on the edges labeled by $k_0^{(j)}$ and $k_{\bar{n}+1}^{(j)}$ interchanged, the discussion for the term (II) is fully analogous to the one of (I).

Summarizing, $\frac{\bar{n}-m_j}{2} + (m_j - a) = \frac{\bar{n}+m_j}{2} - a$ propagators are in total bounded in L^∞ , and $\frac{\bar{n}-m_j}{2} + 2 + a$ in L^1 . In conclusion, we obtain

$$\sup_{\underline{u}_\infty^{(j)}} \int d\underline{u}_1^{(j)} |\text{Amp}_{\hat{J}_{\lambda^2}}(\pi_j(\underline{u}^{(j)}))| < (c\lambda)^{\bar{n}} \varepsilon^{-\frac{\bar{n}+m_j}{2}+a} (\log \frac{1}{\varepsilon})^{\frac{\bar{n}-m_j}{2}+a+2}, \quad (129)$$

as claimed. The cases $n = 0$ and $n = \bar{n}$ are similar, and also yield (129). This can be proved with minor modifications of the arguments explained above, and will not be reiterated. This concludes the proof. \square

Lemma 5.5. *Let $\pi \in \Pi_{s;\bar{n},n}^{(\hat{J}_{\lambda^2})^{conn}}$. We then have the a priori bound*

$$|\text{Amp}_{\hat{J}_{\lambda^2}}(\pi)| < (\log \frac{1}{\varepsilon})^{3s} (c\lambda^2 \varepsilon^{-1} \log \frac{1}{\varepsilon})^{\frac{s\bar{n}}{2}}. \quad (130)$$

Proof. From (107), we have

$$|\text{Amp}_{\hat{J}_{\lambda^2}}(\pi)| \leq \prod_{j=1}^s A_j. \quad (131)$$

Using (131) and Lemma 5.4, we get

$$\begin{aligned} |\text{Amp}_{\hat{J}_{\lambda^2}}(\pi)| &\leq (c\lambda)^{s\bar{n}} \prod_{j=1}^s \varepsilon^{-\frac{\bar{n}+m_j}{2}+a_j} (\log \frac{1}{\varepsilon})^{\frac{\bar{n}-m_j}{2}+a_j+2} \\ &< (c\lambda)^{s\bar{n}} \varepsilon^{-\frac{s\bar{n}+\sum_j m_j}{2}+\sum_j a_j} (\log \frac{1}{\varepsilon})^{\frac{s\bar{n}-\sum_j m_j}{2}+\sum_j a_j+2s}. \end{aligned} \quad (132)$$

We observe that $m = \sum_j m_j \in 2\mathbb{N}_0$ is twice the number of transfer contractions in π , since it counts the number of transfer vertices. Moreover, $\sum_j a_j = \frac{m}{2}$ because to every transfer contraction, we associate one resolvent estimated in L^1 and one estimated in L^∞ , and $\sum_j a_j$ counts those estimated in L^1 . This implies the asserted bound. \square

Next, we estimate the term B in (107), and show that exploiting the connectedness of a pair of particle lines, one gains a factor $\varepsilon^{\frac{1}{5}}$ over the bound $B \leq A_{s-1}A_s$ inferred from Lemma 5.4.

Lemma 5.6. *Assume that the reduced 1-particle lines $\pi_j(\underline{u}^{(j)})$ and $\pi_{j'}(\underline{u}^{(j')})$ have $m_{j;j'}$ common transfer momenta $\underline{u}^{(j;j')}$. Let $\widetilde{\underline{u}}^{(i;j)}$ denote the $m_j + m_{j'} - 2m_{j;j'}$ transfer momenta appearing in either $\underline{u}^{(j)}$ or $\underline{u}^{(j')}$, but not in both. Moreover, assume that $\widehat{\phi}_0$ satisfies the concentration of singularity condition (31). Then,*

$$\begin{aligned} & \sup_{\widetilde{\underline{u}}^{(j;j')}} \int d\underline{u}^{(j;j')} |\text{Amp}_{\widehat{J}_{\lambda^2}}(\pi_j(\underline{u}^{(j)}))| |\text{Amp}_{\widehat{J}_{\lambda^2}}(\pi_{j'}(\underline{u}^{(j')}))| \\ & \leq \lambda^{2\bar{n}} \varepsilon^{\frac{1}{5} - \bar{n} - \frac{m_j + m_{j'}}{2} + m_{j;j'}} (c \log \frac{1}{\varepsilon})^{\bar{n} - \frac{m_j + m_{j'}}{2} + m_{j;j'} + 4} \end{aligned} \quad (133)$$

which improves the corresponding a priori bound by a factor $\varepsilon^{\frac{1}{5}}$.

Proof. To estimate the l.h.s. of (133), we use $L^\infty - L^1$ -bounds in the variables $\underline{u}^{(j;j')}$, with the exception of one transfer momentum, which we denote by u . Thereby, we cut all but one transfer lines between the j -th and the j' -th reduced 1-particle line.

One straightforwardly obtains (133) if it is possible to identify a subgraph in the expression for (133) that corresponds to the "crossing integral"

$$\begin{aligned} & \sup_{\gamma_i \in I} \sup_{k \in \mathbb{T}^3} \left[\int dp_1 dp_2 \frac{1}{|e_\Delta(p) - \gamma_1 - i\varepsilon_1| |e_\Delta(q) - \gamma_2 - i\varepsilon_1|} \right. \\ & \quad \left. \frac{1}{|e_\Delta(p - q + k) - \gamma_3 - i\varepsilon|} \right] \leq c \varepsilon^{-\frac{4}{5}} (\log \frac{1}{\varepsilon})^3, \end{aligned} \quad (134)$$

see Lemma 3.11 in [2]. Here, one of the three resolvents would have been estimated in L^∞ by $\frac{1}{\varepsilon}$ in the a priori bound. There is a gain of a factor $\varepsilon^{\frac{1}{5}}$ because the singularities which contribute most to (134) are concentrated in tubular ε -neighborhoods of level surfaces of e_Δ , whose intersections are of small measure (the curvature of the level surfaces of the energy function e_Δ plays a crucial role for this result).

On each reduced 1-particle line, we identify the contraction structure based on internal deltas, see also Figure 3. As explained in detail in [4, 2], the only possible cases are (we are here omitting the labels j, j' of the reduced 1-particle lines):

- The internal contractions of the reduced 1-particle line define a ladder graph decorated with progressions of immediate recollisions. That is, every internal contraction is either an immediate recollision (a contraction between neighboring internal vertices, possibly with transfer vertices located inbetween), or a rung of the ladder contracting a vertex labeled by $i \leq n$ with a vertex labeled by $i' > n$. For any pair of rung contractions labeled by (i_1, i'_1) and (i_2, i'_2) , one has $i_1 < i_2$, and $i'_1 > i'_2$ (no crossing of rungs). These were denoted "simple graphs" in [4, 2].
- Otherwise, one can identify at least one nesting or crossing subgraph. A pair of internal deltas $\delta(k_{i_1+1} - k_{i_1} + k_{i'_1+1} - k_{i'_1})$ and $\delta(k_{i_2+1} - k_{i_2} + k_{i'_2+1} - k_{i'_2})$ defines a *nesting subgraph* if $i_1 < i_2 < i'_2 < i'_1$, and either $i'_1 \leq n$ or $i_1 > n$. It defines a *crossing subgraph* if $i_1 < i_2 < i'_1 < i'_2$, and either $i'_2 \leq n$ or $i_1 > n$.

In (133), one can identify a crossing subintegral of the form (134) in the following situations:

- One of the reduced 1-particle graphs contains a nesting or crossing subgraph consisting of internal contraction lines, similarly as in [4, 2]. Then, one can completely disconnect the j -th and the j' -th reduced 1-particle line by $L^\infty - L^1$ -estimates in $\underline{u}^{(j;j')}$, and one still gains a factor $\varepsilon^{\frac{1}{5}}$ from (134).
- Both reduced 1-particle subgraphs correspond to ladder graphs with immediate recollision insertions (denoted "simple graphs" in [4, 2]), but there is at least one transfer contraction between the j -th and the j' -th reduced 1-particle line whose ends are located either between rungs of the ladder (that is, not on the left or right of the outermost rung contraction $\delta(k_{i_*+1} - k_{i_*} + k_{i'_*+1} - k_{i'_*})$, where i_* is the smallest, and i'_* is the largest index appearing in any rung contraction on the given reduced 1-particle line) and/or inside an immediate recollision subgraph. The integral over the associated transfer momentum u then produces a subintegral of the form (134), and one gains a factor $\varepsilon^{\frac{1}{5}}$.

The crossing estimate cannot be applied in its basic form (134) when the j -th and the j' -th reduced 1-particle graphs have ladder structure, and every transfer contraction between the j -th and the j' -th reduced 1-particle graph is adjacent to at least one vertex on the left or right of the outermost rung contraction, which is also not located inside an immediate recollision subgraph. Then, the corresponding integrals do not only involve propagators, but also $\hat{\phi}_0$, which itself typically exhibits singularities.

The situation is most difficult to handle if both of the adjacent transfer vertices are of that type. Then, the only subintegral with crossing structure has the form

$$A_\varepsilon := \int d\alpha_1 d\alpha_2 \int dp dq du |\hat{\phi}_0(p)| |\hat{\phi}_0(p+u+k)| |\hat{\phi}_0(q)| |\hat{\phi}_0(q-u)| \frac{1}{|e_\Delta(p) - \alpha_1 - i\varepsilon| |e_\Delta(p+u+k) - \beta_1 + i\varepsilon|} \frac{1}{|e_\Delta(q) - \alpha_2 - i\varepsilon| |e_\Delta(q-u) - \beta_2 + i\varepsilon|} . \quad (135)$$

This expression is obtained from partitioning the integrals on the r.h.s. of (133) in the same way as in the proof of Lemma 5.4 (we recall that on each reduced 1-particle line, one of the energy parameters α_j or β_j is always used to estimate a propagator neighboring to either $\hat{\phi}_0$ or $\bar{\hat{\phi}}_0$ in L^1). In (135), singularities of $|\hat{\phi}_0|$ may overlap with those of the neighboring resolvents; crossing structures then also depend on the singularity structure of $|\hat{\phi}_0|$. If we argue as in the proof of Lemma 5.4, we would use two momentum integrals for $\|\hat{\phi}_0\|_{L^2(\mathbb{T}^3)}^4$, and the remaining integrals (α_1, α_2 , and the third momentum) to bound three resolvents in $L^1(\mathbb{T}^3)$ by $c \log \frac{1}{\varepsilon}$ so that one resolvent is estimated in $L^\infty(\mathbb{T}^3)$ by $\frac{1}{\varepsilon}$. Thereby, one gets

$$A_\varepsilon < c\varepsilon^{-1} (\log \frac{1}{\varepsilon})^3 \|\hat{\phi}_0\|_{L^2(\mathbb{T}^3)}^4 . \quad (136)$$

The remaining terms contributing to the l.h.s. of (134) are estimated in the same way as in the proof of Lemma 5.4 (i.e. by introduction of a spanning tree, and use of

$L^1 - L^\infty$ -bounds on the propagators), whereby one again arrives at the expression for the a priori bound, which is the r.h.s. of (134) without the $\varepsilon^{\frac{1}{5}}$ -factor. We shall not repeat the detailed argument.

To prove (133), we improve (136) by

$$A_\varepsilon \leq c(T) \varepsilon^{-\frac{4}{5}} \left(\log \frac{1}{\varepsilon}\right)^4, \quad (137)$$

where the constant depends only on the macroscopic time $T > 0$. Our proof uses the η -concentration property of the WKB initial data $\widehat{\phi}_0$. We do not know if for general L^2 initial data, or for WKB initial conditions without any restrictions on the phase function S , (136) can be improved.

We recall the concentration of singularity condition (29) - (31), by which

$$\widehat{\phi}_0(k) = f_\infty(k) + f_{sing}(k), \quad (138)$$

where

$$\|f_\infty\|_\infty < c, \quad (139)$$

and

$$\| |f_{crit}| * |f_{crit}| \|_{L^2(\mathbb{T}^3)} = \| |f_{sing}|^\vee \|_{\ell^4(\mathbb{Z}^3)}^2 \leq c' \eta^{\frac{4}{5}} \quad (140)$$

for constants c, c' that are uniform in η .

We observe that A_ε has the form

$$\begin{aligned} A_\varepsilon[g_1, g_2, g_3, g_4] &= \int_{I^2} d\alpha_1 d\alpha_2 \langle \widehat{g}_1 * \widehat{g}_2, \widehat{g}_3 * \widehat{g}_4 \rangle_{L^2(\mathbb{T})} \\ &= \int_{I^2} d\alpha_1 d\alpha_2 \langle g_1 g_2, g_3 g_4 \rangle_{\ell^2(\mathbb{Z}^3)} \\ &= \sum_{r_i \in \{\infty, crit\}} A_\varepsilon[g_{1,r_1}, g_{2,r_2}, g_{3,r_3}, g_{4,r_4}], \end{aligned} \quad (141)$$

where $\widehat{g}_1(p) := \frac{1}{|e_\Delta(p) - \alpha_1 - i\varepsilon|} |\widehat{\phi}_0(p)|$, etc., and where $\widehat{g}_{i,r}$ is obtained from replacing $\widehat{\phi}_0$ by f_r in \widehat{g}_i , for $r \in \{\infty, crit\}$. The corresponding terms can then be bounded as follows.

First of all, if $r_i = \infty$ for $i = 1, \dots, 4$,

$$\begin{aligned} &A_\varepsilon[g_{1,\infty}, g_{2,\infty}, g_{3,\infty}, g_{4,\infty}] \\ &\leq \|f_\infty\|_\infty^4 \sup_{\alpha_j, \beta_j} \int dp dq du \frac{1}{|e_\Delta(p) - \alpha_1 - i\varepsilon| |e_\Delta(p+u) - \beta_1 + i\varepsilon|} \\ &\quad \frac{1}{|e_\Delta(q) - \alpha_2 - i\varepsilon| |e_\Delta(q-u) - \beta_2 + i\varepsilon|} \\ &< c \varepsilon^{-\frac{4}{5}} \left(\log \frac{1}{\varepsilon}\right)^4, \end{aligned} \quad (142)$$

using (134).

If $r_i = \text{crit}$ for one value of i ,

$$\begin{aligned}
& A_\varepsilon[g_{1,\text{crit}}, g_{2,\infty}, g_{3,\infty}, g_{4,\infty}] \\
& \leq \|f_\infty\|_\infty^3 \left[\int dp |f_{\text{crit}}(p)| \right] \\
& \quad \left[\sup_p \frac{1}{e_\Delta(p+u) - \beta_1 + i\varepsilon} \right] \left[\int dq \frac{1}{|e_\Delta(q) - \beta_2 + i\varepsilon|} \right] \\
& \quad \left[\sup_p \int_I d\alpha_1 \frac{1}{|e_\Delta(p) - \alpha_1 - i\varepsilon|} \right] \left[\sup_q \int_I d\alpha_2 \frac{1}{|e_\Delta(q) - \alpha_2 - i\varepsilon|} \right] \\
& < c\varepsilon^{-1} \eta^{\frac{2}{5}} \left(\log \frac{1}{\varepsilon} \right)^4, \tag{143}
\end{aligned}$$

using $\|f_{\text{crit}}\|_{L^1(\mathbb{T}^3)} \leq c\eta^{\frac{2}{5}}$, which follows from

$$\begin{aligned}
\|f_{\text{crit}}\|_{L^1(\mathbb{T}^3)}^2 &= \int dp du |f_{\text{crit}}(p)| |f_{\text{crit}}(u)| \\
&= \int dp du |f_{\text{crit}}(p)| |f_{\text{crit}}(u-p)| \\
&\leq \left(\int du \left[\int dp |f_{\text{crit}}(p)| |f_{\text{crit}}(u-p)| \right]^2 \right)^{\frac{1}{2}} \left(\int du \right)^{\frac{1}{2}} \\
&= \| |f_{\text{crit}}| * |f_{\text{crit}}| \|_{L^2(\mathbb{T}^3)} \\
&\leq c\eta^{\frac{4}{5}}, \tag{144}
\end{aligned}$$

see (140), and where we have used $\text{Vol}(\mathbb{T}^3) = 1$. The remaining cases $r_1 = r_3 = r_4 = \infty$, $r_2 = \text{crit}$, etc., are similar.

If $r_i = \text{crit}$ for two values of i ,

$$\begin{aligned}
& \int_I d\alpha_1 d\alpha_2 A_\varepsilon[g_{1,\text{crit}}, g_{2,\text{crit}}, g_{3,\infty}, g_{4,\infty}] \\
& \leq \|f_\infty\|_\infty^2 \int_{I^2} d\alpha_1 d\alpha_2 \int dp dq du |f_{\text{crit}}(p)| |f_{\text{crit}}(p+u)| \frac{1}{|e_\Delta(p) - \alpha_1 - i\varepsilon|} \\
& \quad \frac{1}{|e_\Delta(p+u) - \beta_1 + i\varepsilon| |e_\Delta(q) - \alpha_2 - i\varepsilon| |e_\Delta(q-u) - \beta_2 + i\varepsilon|} \\
& \leq \|f_\infty\|_\infty^2 \left[\int dp |f_{\text{crit}}(p)| \right] \left[\int du |f_{\text{crit}}(p+u)| \right] \varepsilon^{-1} \\
& \quad \left[\sup_u \int dq \frac{1}{|e_\Delta(q-u) - \beta_2 + i\varepsilon|} \right] \\
& \quad \left[\sup_p \int_I d\alpha_1 \frac{1}{|e_\Delta(p) - \alpha_1 - i\varepsilon|} \right] \left[\sup_q \int_I d\alpha_2 \frac{1}{|e_\Delta(q) - \alpha_2 - i\varepsilon|} \right] \\
& < c\varepsilon^{-1} \eta^{\frac{4}{5}} \left(\log \frac{1}{\varepsilon} \right)^3, \tag{145}
\end{aligned}$$

again using (144). The cases $r_1 = r_3 = \infty$, $r_2 = r_4 = \text{crit}$, etc., are similar.

If $r_i = \text{crit}$ for three values of i ,

$$\begin{aligned}
& \int_I d\alpha_1 d\alpha_2 A_\varepsilon[g_{1,\text{crit}}, g_{2,\text{crit}}, g_{3,\text{crit}}, g_{4,\infty}] \\
& \leq \|f_\infty\|_\infty \int_{I^2} d\alpha_1 d\alpha_2 \int dp dq du |f_{\text{crit}}(p)| |f_{\text{crit}}(p+u)| |f_{\text{crit}}(q)| \\
& \quad \frac{1}{|e_\Delta(p) - \alpha_1 - i\varepsilon| |e_\Delta(p+u) - \beta_1 + i\varepsilon|} \\
& \quad \frac{1}{|e_\Delta(q) - \alpha_2 - i\varepsilon| |e_\Delta(q-u) - \beta_2 + i\varepsilon|} \\
& \leq \|f_\infty\|_\infty \|f_{\text{crit}}\|_{L^1(\mathbb{T}^3)}^3 \varepsilon^{-2} \\
& \quad \left[\sup_p \int_I d\alpha_1 \frac{1}{|e_\Delta(p) - \alpha_1 - i\varepsilon|} \right] \left[\sup_q \int_I d\alpha_2 \frac{1}{|e_\Delta(q) - \alpha_2 - i\varepsilon|} \right] \\
& \leq c \varepsilon^{-2} \eta^{\frac{6}{5}} (\log \frac{1}{\varepsilon})^2 \tag{146}
\end{aligned}$$

using (144). The remaining cases are similar.

Finally, if $r_i = \text{crit}$ for all values of i ,

$$\begin{aligned}
& \int_I d\alpha_1 d\alpha_2 A_\varepsilon[g_{1,\text{crit}}, g_{2,\text{crit}}, g_{3,\text{crit}}, g_{4,\text{crit}}] \\
& \leq \int_{I^2} d\alpha_1 d\alpha_2 \int dp dq du |f_{\text{crit}}(p)| |f_{\text{crit}}(p+u)| |f_{\text{crit}}(q)| |f_{\text{crit}}(q-u)| \\
& \quad \frac{1}{|e_\Delta(p) - \alpha_1 - i\varepsilon| |e_\Delta(p+u) - \beta_1 + i\varepsilon|} \\
& \quad \frac{1}{|e_\Delta(q) - \alpha_2 - i\varepsilon| |e_\Delta(q-u) - \beta_2 + i\varepsilon|} \\
& \leq \varepsilon^{-2} \| |f_{\text{crit}}| * |f_{\text{crit}}| \|_{L^2(\mathbb{T}^3)}^2 \\
& \quad \left[\sup_p \int_I d\alpha_1 \frac{1}{|e_\Delta(p) - \alpha_1 - i\varepsilon|} \right] \left[\sup_q \int_I d\alpha_2 \frac{1}{|e_\Delta(q) - \alpha_2 - i\varepsilon|} \right] \\
& < c \varepsilon^{-2} \eta^{\frac{8}{5}} (\log \frac{1}{\varepsilon})^2, \tag{147}
\end{aligned}$$

using (140).

We recall that $\eta = \lambda^2 = T\varepsilon$, where $\varepsilon = \frac{1}{t}$ is the inverse microscopic time, and $T = \lambda^2 t$ is the macroscopic time.

Collecting the estimates on (141) derived above, we find that for any $T > 0$, there is a constant $c(T) < \frac{c}{T^2}$ such that

$$A_\varepsilon \leq c(T) \varepsilon^{-\frac{4}{5}} (\log \frac{1}{\varepsilon})^4. \tag{148}$$

This estimate improves (136) by a factor $\varepsilon^{\frac{1}{5}}$, as claimed, and establishes (133). \square

Using the arguments used in the proof of Lemma 5.5, one hereby also establishes Lemma 5.2.

Moreover, we find the following bounds.

Lemma 5.7. *Let $r \in 2\mathbb{N}$, and let $\Pi_{r;\bar{n},n}^{(\hat{J}_{\lambda^2})^{2-\text{conn}}}, \Pi_{r;\bar{n},n}^{(\hat{J}_{\lambda^2})^{n-d}} \subset \Pi_{r;\bar{n},n}^{(\hat{J}_{\lambda^2})}$ denote the subclasses of 2-connected and non-disconnected graphs, respectively. Then, for every $T = \lambda^2 \varepsilon^{-1} > 0$, there exists a finite constant $c = c(T)$ such that*

$$\sum_{\pi \in \Pi_{r;\bar{n},n}^{(\hat{J}_{\lambda^2})^{2-\text{conn}}}} |\text{Amp}_{\hat{J}_{\lambda^2}}(\pi)| \leq (r\bar{n})! \varepsilon^{\frac{1}{5}} \left(\log \frac{1}{\varepsilon}\right)^{3r} (c\lambda^2 \varepsilon^{-1} \log \frac{1}{\varepsilon})^{\frac{r\bar{n}}{2}}. \quad (149)$$

and

$$\sum_{\pi \in \Pi_{r;\bar{n},n}^{(\hat{J}_{\lambda^2})^{n-d}}} |\text{Amp}_{\hat{J}_{\lambda^2}}(\pi)| \leq (r\bar{n})! \varepsilon^{\frac{1}{5}} \left(\log \frac{1}{\varepsilon}\right)^{3r} (c\lambda^2 \varepsilon^{-1} \log \frac{1}{\varepsilon})^{\frac{r\bar{n}}{2}}. \quad (150)$$

Proof. This follows immediately from Lemma 5.2 and the fact that $\Pi_{r;\bar{n},n}^{(\hat{J}_{\lambda^2})}$ contains no more than $(r\bar{n})! 2^{r\bar{n}}$ graphs. \square

Lemma 5.8. *For any fixed $r \geq 2$, $r \in 2\mathbb{N}$, $n \leq N$, and $T > 0$, there exists a finite constant $c = c(T)$ such that*

$$\left(\mathbb{E} \left[\|\phi_{n,t}\|_2^{2r} \right] \right)^{\frac{1}{r}} \leq ((2nr)!)^{\frac{1}{r}} \left(\log \frac{1}{\varepsilon}\right)^3 (c\lambda^2 \varepsilon^{-1} \log \frac{1}{\varepsilon})^n. \quad (151)$$

Proof. This is proved in the same way as the a priori bound of Lemma 5.5. The only modification is that the \hat{J}_{λ^2} -delta is replaced by $\delta(k_n^{(j)} - k_{n+1}^{(j)})$ on every particle line. We note that the expansion for (151) contains disconnected graphs. \square

6. PROOF OF LEMMA 4.2

Based on the previous discussion, is straightforward to see that

$$\begin{aligned} & \mathbb{E} \left[\|\phi_{n,N,\theta_{m-1}}(\theta_m)\|_2^{2r} \right] \\ &= \frac{e^{2r(1+\theta_{m-1}\varepsilon)}}{(2\pi)^{2r}} \int_{(I \times \bar{I})^r} \prod_{j=1}^r d\alpha_j d\beta_j e^{-i\theta_m \sum_{j=1}^r (-1)^j (\alpha_j - \beta_j)} \\ & \quad \int_{(\mathbb{T}^3)^{(\bar{n}+2)r}} \left[\prod_{j=1}^r d\underline{k}^{(j)} \delta(k_n^{(j)} - k_{n+1}^{(j)}) \right] \lambda^{r\bar{n}} \mathbb{E} \left[\prod_{j=1}^r U^{(j)}[\underline{k}^{(j)}] \right] \\ & \quad \left[\prod_{j=1}^r K_{n,N,\kappa}^{(j)}[\underline{k}^{(j)}, \alpha_j, \beta_j, \varepsilon] \right] \widehat{\phi}_0^{(j)}(k_0^{(j)}) \overline{\widehat{\phi}_0^{(j)}(k_{\bar{n}+1}^{(j)})}. \end{aligned} \quad (152)$$

(using $(\theta_m - \theta_{m-1})\kappa\varepsilon = 1$) where

$$K_{n,N,\kappa}^{(j)} := \frac{1}{(e_{\Delta}(k_n^{(j)}) - \alpha_j - i\varepsilon_j)(e_{\Delta}(k_{n+1}^{(j)}) - \beta_j + i\varepsilon_j)} \widetilde{K}_{n,N,\kappa}^{(j)}, \quad (153)$$

and

$$\begin{aligned} & \tilde{K}_{n,N,\kappa}^{(j)}[\underline{k}^{(j)}, \alpha_j, \beta_j, \varepsilon] \\ &:= \prod_{\ell_1=0}^{n-N} \frac{1}{e_{\Delta}(k_{\ell_1}^{(j)}) - \alpha_j - i\kappa\varepsilon_j} \prod_{\ell_2=n-N+1}^{n-1} \frac{1}{e_{\Delta}(k_{\ell_2}^{(j)}) - \alpha_j - i\varepsilon_j} \\ & \quad \prod_{\ell_3=n+2}^{n+N} \frac{1}{e_{\Delta}(k_{\ell_3}^{(j)}) - \beta_j + i\varepsilon_j} \prod_{\ell_4=n+N+1}^{2n+1} \frac{1}{e_{\Delta}(k_{\ell_4}^{(j)}) - \beta_j + i\kappa\varepsilon_j}, \end{aligned} \quad (154)$$

see also (85) and the discussion following (80). We refer to $\delta(k_n^{(j)} - k_{n+1}^{(j)})$, which replaces the \hat{J}_{λ^2} -delta, as the " L^2 -delta", since it is responsible for the L^2 -inner product on the left hand side of (152). The expression (152) can be estimated in the same way as the integrals (87) considered above, however, we are now considering the full instead of the non-disconnected expectation.

As before, $\mathbb{E}[\prod_{j=1}^r U^{(j)}[\underline{k}^{(j)}]]$ in (152) decomposes into a sum of products of delta distributions, which we represent by Feynman graphs. By the notational conventions introduced after (87), we have $\bar{n} = 2n$. We let $\Pi_{r;\bar{n},n}$ denote the set of graphs on r particle lines, each containing \bar{n} vertices from copies of the random potential \hat{V}_{ω} , and with the L^2 -delta located between the n -th and the $n+1$ -st \hat{V}_{ω} -vertex. For $\pi \in \Pi_{r;\bar{n},n}$, let $\text{Amp}_{\delta}(\pi)$ denote the amplitude corresponding to the graph π , given by the integral obtained from replacing $\mathbb{E}[\prod_{j=1}^r U^{(j)}[\underline{k}^{(j)}]]$ in (152) by $\delta_{\pi}(\underline{k}^{(1)}, \dots, \underline{k}^{(r)})$ (the product of delta distributions corresponding to the contraction graph π). The subscript in Amp_{δ} implies that instead of \hat{J}_{λ^2} as before, we now have the L^2 -delta at the distinguished vertex.

Let $\Pi_{r;\bar{n},n}^{conn}$ denote the subclass of $\Pi_{r;\bar{n},n}$ of completely connected graphs.

Lemma 6.1. *Let $s \geq 2$, $s \in \mathbb{N}$, and let $\pi \in \Pi_{s;2n,n}^{conn}$ (that is, $\bar{n} = 2n$) be a completely connected graph. Then, for every $T = \lambda^2 \varepsilon^{-1} > 0$, there exists a finite constant $c = c(T)$ such that*

$$|\text{Amp}_{\delta}(\pi)| \leq \varepsilon^{\frac{1}{5}} (\log \frac{1}{\varepsilon})^{3s} (c\lambda^2 \varepsilon^{-1} \log \frac{1}{\varepsilon})^{sn}. \quad (155)$$

Proof. The proof is completely analogous to the one given for Lemma 5.2 (using $\frac{1}{\kappa\varepsilon} \leq \frac{1}{\varepsilon}$), and will not be reiterated here. \square

In contrast to the situation in Lemma 5.2, the expectation in (152) contains completely disconnected graphs, which satisfy

$$\sum_{\pi \in \Pi_{r;\bar{n},n}^{disc}} |\text{Amp}_{\delta}(\pi)| \leq \left(\sum_{\pi \in \Pi_{1;\bar{n},n}^{conn}} |\text{Amp}_{\delta}(\pi)| \right)^r. \quad (156)$$

We invoke the following bound from [2] (the continuum version is proved in [4]).

Lemma 6.2. *Let $\bar{n} = 2n$. Then, for a constant c independent of T ,*

$$\sum_{\pi \in \Pi_{1;\bar{n},n}^{conn}} |\text{Amp}_{\delta}(\pi)| \leq \frac{(c\lambda^2 \varepsilon^{-1})^n}{\sqrt{n!}} + (n!) \varepsilon^{\frac{1}{5}} (\log \frac{1}{\varepsilon})^3 (c\lambda^2 \varepsilon^{-1} \log \frac{1}{\varepsilon})^n. \quad (157)$$

The term $\frac{(c\lambda^2\varepsilon^{-1})^n}{\sqrt{n!}}$ bounds the contribution from decorated ladder diagrams, while the term that carries an additional $\varepsilon^{\frac{1}{5}}$ -factor is obtained from crossing and nesting type subgraphs. The proof of (157) is presented in detail in [2] and [4]. The number of non-ladder graphs is bounded by $n!2^n$, hence the factor $n!$.

The sum over non-disconnected graphs can be estimated by the same bound as in Lemma 5.7. The result is formulated in the following lemma.

Lemma 6.3. *Let $r \in 2\mathbb{N}$, $\bar{n} = 2n$, and $\Pi_{r;\bar{n},n}^{n-d} \subset \Pi_{r;\bar{n},n}$ denote the subclass of non-disconnected graphs. Then, for every $T > 0$, there exists a finite constant $c = c(T)$ such that*

$$\sum_{\pi \in \Pi_{r;\bar{n},n}^{n-d}} |\text{Amp}_\delta(\pi)| \leq (r\bar{n})! \varepsilon^{\frac{1}{5}} \left(\log \frac{1}{\varepsilon}\right)^{3r} (c\lambda^2\varepsilon^{-1} \log \frac{1}{\varepsilon})^{\frac{r\bar{n}}{2}}. \quad (158)$$

Combining (157) with Lemma 6.3, and applying the Minkowski inequality, the statement of Lemma 4.2 follows straightforwardly.

7. PROOF OF LEMMA 4.3

For $r \in 2\mathbb{N}$, $s \in [\theta_{m-1}, \theta_m]$, $\theta_m - \theta_{m-1} = \frac{t}{\kappa}$, $n = 4N$, and $\bar{n} = 8N$, one gets

$$\begin{aligned} & \mathbb{E}[\|\tilde{\phi}_{4N,N,\theta_{m-1}}(s)\|_2^{2r}] \\ &= \frac{e^{2r((s-\theta_m)\kappa+\theta_m)\varepsilon}}{(2\pi)^{2r}} \int_{(I \times \bar{I})^r} \prod_{j=1}^r d\alpha_j d\beta_j e^{-is \sum_{j=1}^r (-1)^j (\alpha_j - \beta_j)} \\ & \quad \int_{(\mathbb{T}^3)^{(\bar{n}+2)r}} \left[\prod_{j=1}^r d\underline{k}^{(j)} \delta(k_{4N}^{(j)} - k_{4N+1}^{(j)}) \right] \lambda^{r\bar{n}} \mathbb{E} \left[\prod_{j=1}^r U^{(j)}[\underline{k}^{(j)}] \right] \\ & \quad \left[\prod_{j=1}^r \tilde{K}_{4N,N,\kappa}^{(j)}[\underline{k}^{(j)}, \alpha_j, \beta_j, \varepsilon] \right] \widehat{\phi}_0^{(j)}(k_0^{(j)}) \overline{\widehat{\phi}_0^{(j)}(k_{8N+1}^{(j)})}, \quad (159) \end{aligned}$$

where $\varepsilon = \frac{1}{t}$. The notations are the same as in the proof of Lemma 4.2. See (154) for the definition of $\tilde{K}_{4N,N,\kappa}^{(j)}$. We note that here, the propagators on each particle line previously labeled by n and $n+1$ are absent, since we are considering $\tilde{\phi}_{4N,N,\theta_{m-1}}(\theta_m)$ instead of $\phi_{n,N,\theta_{j-m}}(\theta_m)$, see (46).

Let $\Pi_{r;8N,4N}^{conn}$ denote the subset of $\Pi_{r;8N,4N}$ of completely connected graphs.

Lemma 7.1. *Let $r \geq 1$, $r \in \mathbb{N}$, and let $\pi \in \Pi_{r;8N,4N}^{conn}$ be a completely connected graph. Then, for every $T = \lambda^2\varepsilon^{-1} > 0$, there exists a finite constant $c = c(T)$ such that*

$$|\text{Amp}_\delta(\pi)| \leq \kappa^{-2rN} \left(\log \frac{1}{\varepsilon}\right)^{3r} (c\lambda^2\varepsilon^{-1} \log \frac{1}{\varepsilon})^{\frac{r\bar{n}}{2}}. \quad (160)$$

Proof. We modify the proof of Lemma 5.5 in the following manner. We observe that (152) contains $r(6N+2)$ propagators with imaginary parts $\pm i\kappa\varepsilon$, and $2rN$ propagators with imaginary parts $\pm i\varepsilon$. In the proof of Lemma 5.5, $4rN$ out of all propagators were estimated in L^∞ , while the rest was estimated in L^1 . The

fact that there are two propagators less per reduced 1-particle line leads to an improvement over the estimates of Lemma 5.2 which we, however, do not need to exploit.

Carrying out the same arguments line by line, we estimate $4rN$ out of all propagators in (159) in L^∞ . There are $2rN$ propagators carrying an imaginary part $\pm i\varepsilon$ in the denominator. By the pigeonhole principle, at least $2rN$ propagators bounded in L^∞ have a denominator with an imaginary part $\pm i\kappa\varepsilon$. From each of those, one obtains an improvement of the a priori bound in Lemma 5.5 by a factor κ^{-1} . This is because all propagators estimated in L^∞ in Lemma 5.5 were bounded by $\frac{1}{\varepsilon}$. In total, one gains a factor of at least κ^{-2rN} over the estimate in Lemma 5.5.

For a more detailed exposition of arguments concerning the time partitioning method, we refer to [4]. \square

Lemma 7.2. *Let $r \in 2\mathbb{N}$. Then, for every $T > 0$, there exists a finite constant $c = c(T)$ such that*

$$\sum_{\pi \in \Pi_{r;8N,4N}} |\text{Amp}_\delta(\pi)| \leq (4rN)! \kappa^{-2rN} (\log \frac{1}{\varepsilon})^{3r} (c\lambda^2 \varepsilon^{-1} \log \frac{1}{\varepsilon})^{4rN}. \quad (161)$$

Proof. Let $\pi \in \Pi_{r;8N,4N}$ have m connectivity components, and let π comprise s_1, \dots, s_m particle lines, where $\sum_{l=1}^m s_l = r$. Then,

$$\begin{aligned} |\text{Amp}_\delta(\pi)| &\leq \kappa^{-2N \sum_{l=1}^m s_l} (\log \frac{1}{\varepsilon})^{3 \sum_{l=1}^m s_l} (c\varepsilon^{-1} \lambda^2 \log \frac{1}{\varepsilon})^{\frac{\bar{n} \sum_{l=1}^m s_l}{2}} \\ &\leq \kappa^{-2rN} (\log \frac{1}{\varepsilon})^{3r} (c\varepsilon^{-1} \lambda^2 \log \frac{1}{\varepsilon})^{4rN}. \end{aligned} \quad (162)$$

Moreover, the number of elements of $\Pi_{r;8N,4N}$ is bounded by $(4rN)! 2^{4rN}$. \square

The corresponding sum over disconnected graphs can be estimated by the bound in Lemma 6.3. This proves Lemma 4.3.

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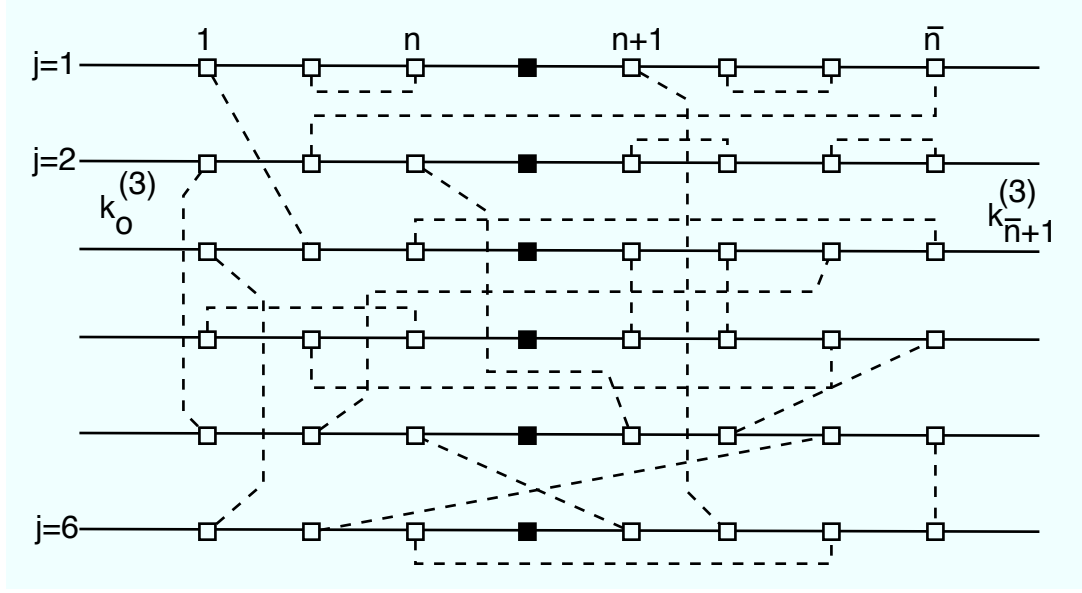


Figure 1. A (completely connected) contraction graph for the case $r = 6$, $n = 3$, $\bar{n} = 7$. The \hat{J}_{λ^2} -vertices are drawn in black, while the \hat{V}_{ω} -vertices are shown in white. The r particle lines are solid, while the lines corresponding to contractions of pairings of random potentials are dashed. For $j = 3$ in the notation of (87), the momenta $k_0^{(3)}$ and $k_{\bar{n}+1}^{(3)}$ are written above the corresponding propagator lines.

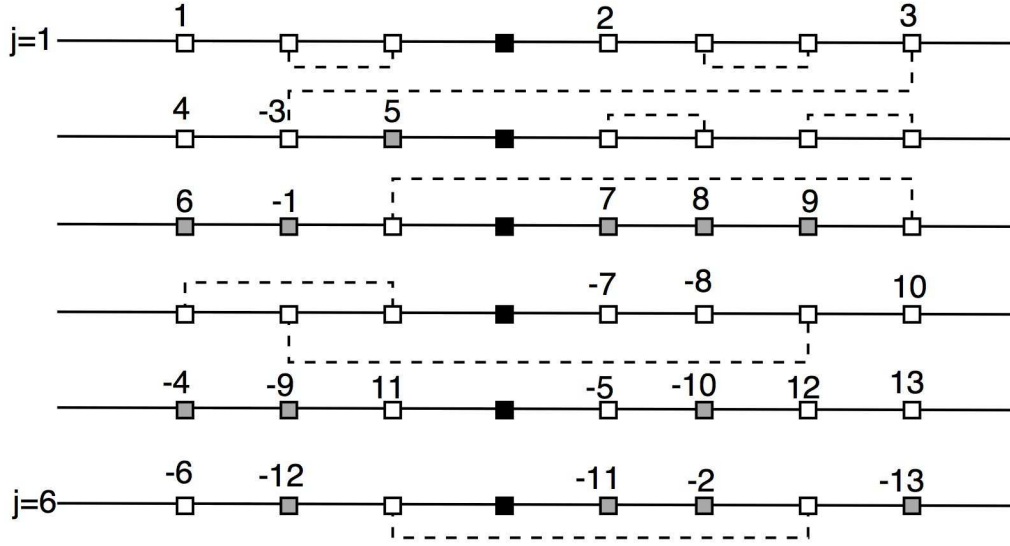


Figure 2. The decomposition of the graph π in Figure 1 into reduced 1-particle lines, with the exception of the particle lines labeled by $j = 1$ and $j = 2$. A numbered vertex with label ℓ accounts for a reduced transfer delta carrying the transfer momentum u_ℓ , and a label $-\ell$ accounts for one carrying a transfer momentum $-u_\ell$. In this example, unfilled numbered transfer vertices carry transfer momenta used for L^∞ -bounds in (131), while shaded transfer vertices carry transfer momenta used for L^1 -bounds.

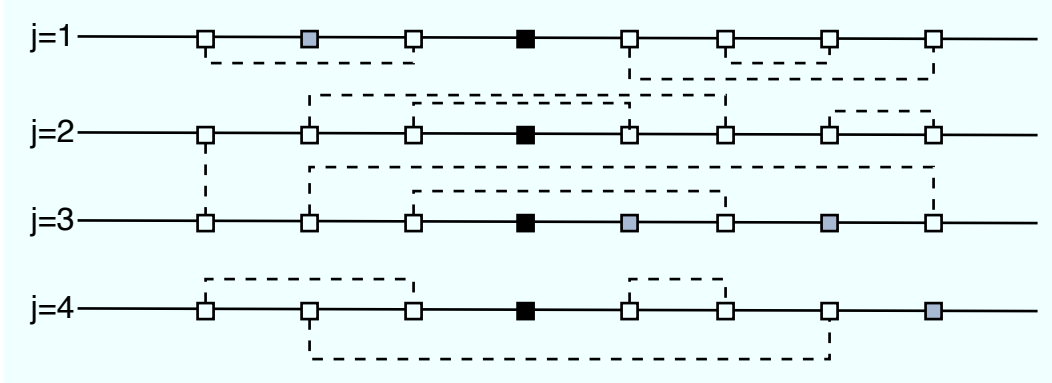


Figure 3. An example unrelated to that in Figures 1 and 2. Here, all reduced transfer vertices are shaded, while the unreduced transfer vertices and all internal vertices are unfilled. The reduced 1-particle line with $j = 1$ contains an immediate recollision with a reduced transfer vertex insertion, and a nesting subgraph. The reduced 1-particle lines with $j = 2, 3$ define a ladder diagram with two rungs each, and decorated by a immediate recollision, and connected by a transfer contraction line. The reduced 1-particle line with $j = 4$ contains a crossing subgraph.

DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY, FINE HALL, WASHINGTON ROAD, PRINCETON, NJ 08544, U.S.A.

E-mail address: `tc@math.princeton.edu`